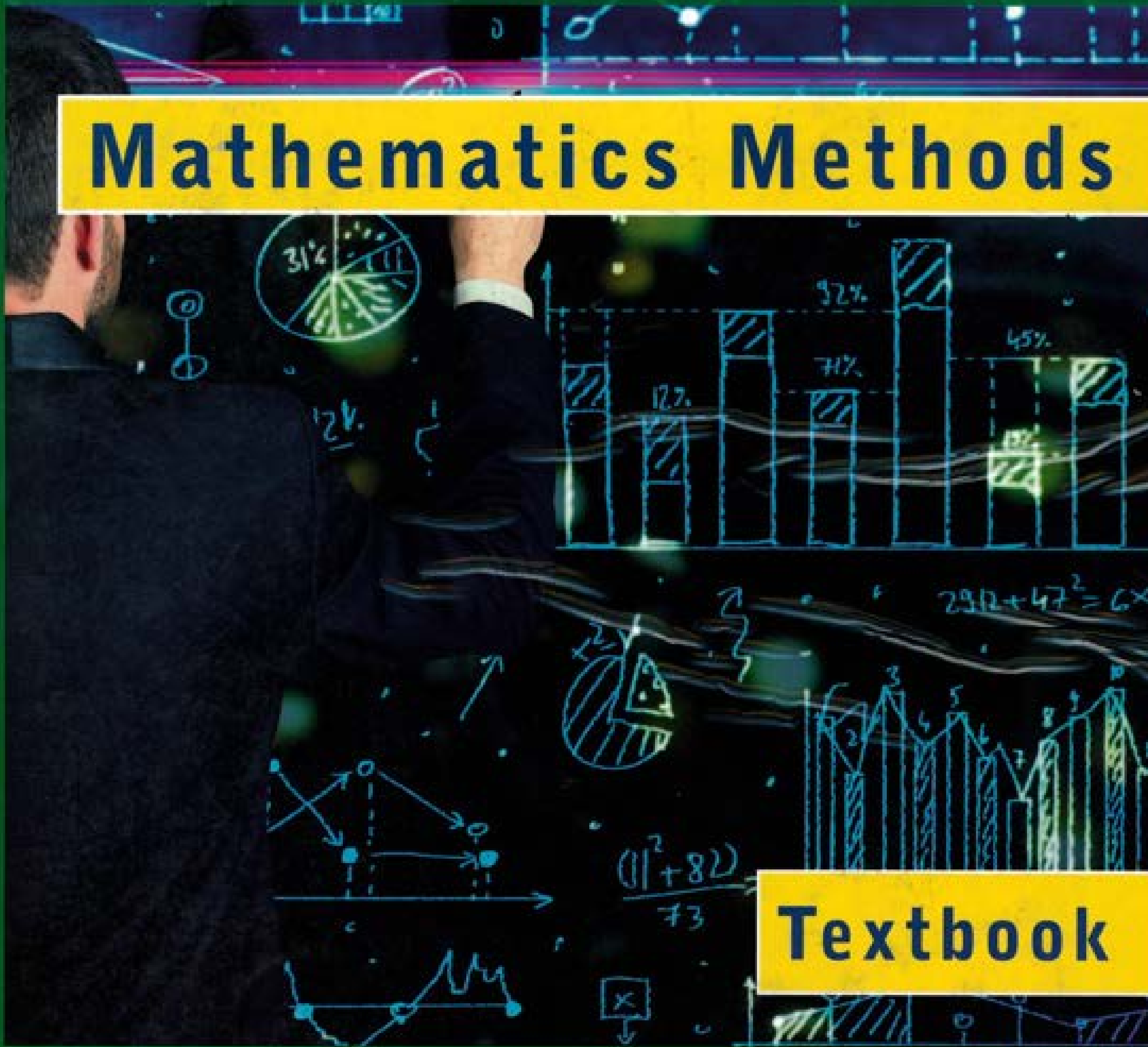
Year 12
Units 3 & 4

ATAR Course
Australian Curriculum

Mathematics Methods



Textbook

O T Lee

ACADEMIC TASK FORCE

Contents

01 Exponential Functions

1.1 Euler's Number *e*

- In this section we will introduce a special number called Euler's number denoted by the lower case letter *e*.

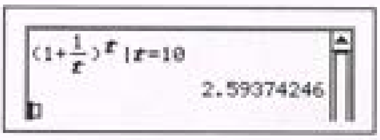
Hands On Task 1.1

In this task, we will explore how Euler's number *e* emerges from two expressions.

1. Consider the expression $\left(1+\frac{1}{t}\right)^t$ for $t > 0$.

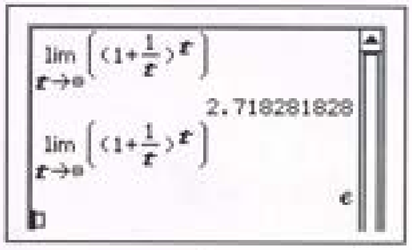
   Calculate the value of this expression for:

   (a) $t = 10$ (b) $t = 100$ (c) $t = 1\,000$
   (d) $t = 10\,000$ (e) $t = 100\,000$ (f) $t = 1\,000\,000$
   (g) $t = 10^8$ (h) $t = 10^{10}$ (i) $t = 10^{12}$



2. Hence, find the value of $\left(1+\frac{1}{t}\right)^t$ as $t \to \infty$ (as t becomes very large).

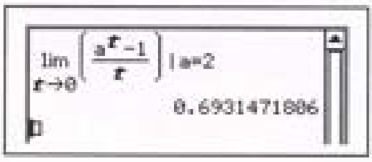
   This limit is a special type of irrational number and is called a transcendental number and rounded to 30 decimal places is
   2.718 281 828 459 045 235 360 287 471 357
   This number is called *Euler's number* (after the Swiss mathematician Leonhard Euler) and is denoted *e*.



3. Consider now the expression $\lim\limits_{t\to 0}\left(\frac{a^t-1}{t}\right)$ where $a > 0$.

   Calculate the value of this expression for:

   (a) $a = 2$ (b) $a = 2.5$ (c) $a = 2.6$
   (d) $a = 2.7$ (e) $a = 2.71$ (f) $a = 2.72$
   (g) $a = 2.8$ (h) $a = 2.9$ (i) $a = 3$



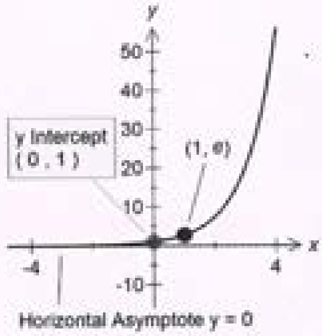
4. Hence, find the value of $\lim\limits_{t\to 0}\left(\frac{a^t-1}{t}\right)$ as $a \to e$ (as a becomes Euler's number).

△ Summary

- Euler's number, e, is the limit to which $\left(1+\frac{1}{t}\right)^t$ tends as $t \to \infty$.
- The expression $\lim_{t\to 0}\left(\frac{a^t - 1}{t}\right) = 1$ as $a \to e$.

1.1.1 The exponential function

- Exponential functions take the form $y = a^x$, where $a > 0$.
- When $a = e$, the function $y = e^x$ is commonly referred to as *the exponential function.*
- $y = 2^x$ is *an* exponential function while $y = e^x$ is *the* exponential function.
- The sketch of $y = e^x$ is given in the accompanying diagram. The obvious points are (0, 1) and $(1, e) \approx (1, 2.7)$. The horizontal asymptote has equation $y = 0$.



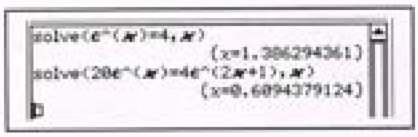
Example 1.1

Use an appropriate method to solve (to 4 D.P.): (a) $e^x = 4$ (b) $20\,e^x = 4\,e^{2x+1}$.

Solution:

(a) $e^x = 4 \quad \Rightarrow x = 1.3863$

(b) $20\,e^x = 4\,e^{2x+1} \quad \Rightarrow \quad x = 0.6094$



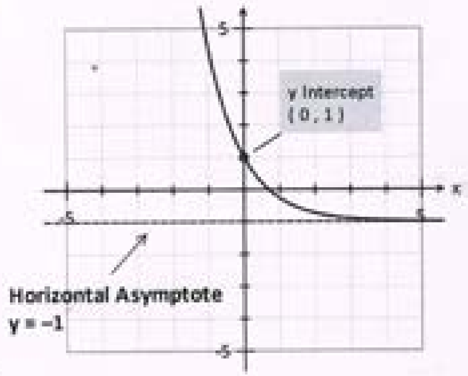
Example 1.2

Without the use of a calculator sketch the curve with equation $y = 2\,e^{-x} - 1$.

Solution:

In sketching the graph of exponential functions:

- *Firstly identify the horizontal asymptote.*
  The $y = e^x$ curve has been translated one unit "downwards". Hence, horizontal asymptote is now $y = -1$.
- *Then identify at least 2 other points.*
  When $x = 0$, $y = 1$.
  When $x = -1$, $y = 2e - 1 \approx 2 \times 2.7 - 1 \approx 4.4$

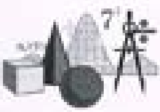


Exercise 1.1

1. Use an appropriate method to solve for x correct to 4 decimal places where appropriate:
   (a) $e^{x} = 1/e$ (b) $e^{x} = 4$ (c) $e^{-0.04x} = 5$
   (d) $e^{1.2x} = 10$ (e) $10\,e^{x} = 5\,e^{1.1x}$ (f) $100\,e^{-0.05x} = 50\,e^{-0.01x}$

2. Sketch each of the following. Indicate at least two obvious points. State the equation of the horizontal asymptote.
   (a) $y = e^{x}$ (b) $y = e^{-x}$ (c) $y = e^{0.5x}$
   (d) $y = 1 + e^{-0.5x}$ (e) $y = 5 - e^{x}$ (f) $y = -2 + e^{-x}$

1.2 Growth and Decay I



Hands On Task 1.2

In this task, we will explore a function that models continuous growth and decay.

A colony of bacteria is being grown in a culture. The initial number of bacteria is 100.
Let the number of bacteria at the end of the nth day be P_n.

1. The colony grows in such a way that the number of bacteria at the end of each day is 10% more than the number of bacteria at the end of the previous day.
   (a) Verify that $P_n = 100(1.1)^{n}$.
   (b) Given that $a^{0.1} = 1.1$, verify that $a \approx 2.593\,742\,460$.

2. The colony grows in such a way that the number of bacteria at the end of each hour is $\left(\frac{10}{24}\right)$% more than the number of bacteria at the end of the previous hour.
   (a) Verify that $P_n = 100(1.0041\dot{6})^{24n}$. (b) Given that $a^{0.1} = 1.00416^{24}$, find a.

3. The colony grows in such a way that the number of bacteria at the end of each minute is $\left(\frac{10}{1440}\right)$% more than the number of bacteria at the end of the previous minute.
   (a) Verify that $P_n = 100(1.000\,069\,\dot{4})^{1440n}$.
   (b) Given that $a^{0.1} = 1.000\,069\,4^{1440}$, find a.

4. The colony grows in such a way that the number of bacteria at the end of each second is $\left(\frac{10}{86400}\right)$% more than the number of bacteria at the end of the previous second.

   (a) Verify that $P_n = 100(1.000\ 001\ 157)^{86400n}$.

   (b) Given that $a^{0.1} = 1.000\ 001\ 157^{86400}$, find a.

5. Use the calculated values of a to complete the following table.

Population count by:	P_n	$P_n = 100(a)^{0.1n}$
days	$P_n = 100(1.1)^n$	$P_n = 100(2.593\ 742\ 460)^{0.1n}$
hours	$P_n = 100(1.0041\dot{6})^{24n}$	
minutes	$P_n = 100(1.000069\dot{4})^{1440n}$	
seconds	$P_n = 100(1.000\ 001\ 157)^{86400n}$	

6. Comment on the value of a as the time interval between the "population count" decreases.

Special Notes:

- In each of the above cases, the rate of increase of the bacteria population was the same, the equivalent of 10% per day.
- The population count was taken over different time intervals. As the time interval decreased, from day to hour to minute to second, the value of a converges to a limit.

Population count by days	$a = 2.5937424601$
Population count by hours	$a = 2.708321497534$
Population count by minutes	$a = 2.71644848697032$
Population count by seconds	$a = 2.71732359027839$

- We could have reduced the time period to milliseconds, microseconds, etc. However, this would exhaust the capabilities of the calculators used.
- Mathematically, it can be shown that as the time period is reduced to a state where the population count is taken *continuously* (every instant), $a \to e$.
- Hence, if the population count were to be taken continuously, for a population with an initial value of 100 and growing at a continuous rate of 10%, $P_n = 100e^{0.1n}$.

△ Summary

- If a population *grows continuously* (exponentially) at a rate of $100k\%$ per year, the population at any time t is given by $P = P_0 e^{kt}$.
  P_0 is the population when $t = 0$, the initial population.
- If a population *decays continuously* (exponentially) at a rate of $100k\%$ per year, the population at any time t is given by $P = P_0 e^{-kt}$.

Example 1.3

A colony of whales grows exponentially (continuously) at a rate of 5% per year.
In January 2010, there were 3 000 of these whales.
(a) Find an expression for the whale population t years after January 2010.
(b) Find when the whale population doubles.
(c) Discuss the validity of this mathematical model in predicting the future whale population.

Solution:

(a) $P = 3\,000e^{0.05t}$, where $t \geq 0$

(b) When the whale population doubles, $P = 6\,000$.

Hence $\quad 3\,000\, e^{0.05t} = 6\,000$

$\Rightarrow \quad t = 13.86$

Therefore, the whale population will double after 13.86 years, that is in the year 2023.

(c) This models assumes that there is no upper limit to the whale population. However, in reality, the population will eventually peak and then decline because the food source will eventually get depleted.

Note:

- *The time taken for the whale population to double is called the doubling time. No matter what the original population is, the time taken for the population to double is always a constant.*

Example 1.4

The amount R, in grams, of a radioactive substance X remaining at time t years is given by $R = 100e^{-0.01t}$, $t \geq 0$.
(a) Find the initial amount of X. (b) Find the continuous rate of decay for X.
(c) Find the amount of X that has decayed after 100 years.
(d) Find how long it will take for the amount of X to be halved.

Solution:

(a) Initial amount of X is 100 g.

(b) The continuous rate of decay of X is $0.01 \times 100 = 1\%$ per year.

(c) When $t = 100$, the amount remaining $R = 100e^{-0.01(100)} = 36.79$ g.
Therefore, the amount that has decayed $= 100 - 36.79 = 63.2$ g.

(d) When the amount of X is halved, $R = 50$.
Hence, $100e^{-0.01t} = 50$
$\Rightarrow t = 69.3$
Hence, it will take 69.3 years for the amount of X to be halved.

Notes:

- *The time taken for X to be halved is called its half-life and is constant for whatever amount of X is present.*

Example 1.5

In a certain outback area, the population of bilbys (an Australian marsupial), t years after 2000, is modelled by $P = 350e^{0.02t}$, while the corresponding population of feral cats in the same region, t years after 2000, is modelled by $P = 200e^{0.05t}$. When will the number of feral cats first outnumber the bilbys.

Solution:

When number of feral cats = number of bilbys

$$200e^{0.05t} = 350e^{0.02t}$$
$$\Rightarrow t = 18.65$$

Hence, the feral cats will outnumber the bilbys 18.65 years after 2000, that is, in the year 2019.

Exercise 1.2

1. Australia's population, (in millions), t years after 2004 is modelled by $P = 20e^{0.0189t}$.
   (a) State the population of Australia in 2004.
   (b) State the continuous growth/decay rate used in this model.
   (c) Find Australia's population 25 years after 2004
   (d) Find how long it will take for Australia's population to reach 30 million.
   (e) Find how long it will take for Australia's population to double.

2. The population of a colony of feral goats grows continuously at a rate of 9% per year. The number of feral goats at the start of a study is 650.
   (a) Find an expression to model P, the number of feral goats, t years into the study.
   (b) Find the feral goat population after 4 years.
   (c) Find the time taken for the feral goat population to reach 1 000.
   (d) Find the time taken for colony to quadruple in size.

3. The population of a colony of koalas is being monitored. The population P, after t years is modelled by $P = 4\,000e^{-0.001t}$.
   (a) Find the initial population of this colony.
   (b) Find the exponential growth/decay rate of this colony.
   (c) Find the population after 5 years.
   (d) Find how long it will take for the population to be halved.

4. A radioactive substance S, decays at a continuous rate of 1% per year. Initially, there was 50 g of this substance.
   (a) Find an expression for A, the amount of S, left after t years.
   (b) Find the amount of S that is left after 100 years.
   (c) Find the amount of S that has decayed after 50 years.
   (d) Find the time it will take for the amount of S to drop to 1 g.

5. The population of the United Kingdom (UK) at the start of 1981 was estimated to be 56 million. Its continuous growth rate was estimated to be 0.2% per year. The population of Papua New Guinea (PNG) was estimated to be 3.1 million at the start of 1981 with a continuous growth rate of 2.0% per year.
   (a) Find the population of PNG at the start of the year 2000.
   (b) Find when the population of PNG, first exceeds that of the UK.
   (c) Comment on the validity of your answer in (b).

6. The estimated population and continuous growth rate of China in 1981 were 991.3 million and 1.0% per year respectively. The corresponding figures for India in 1981 were 690.2 million and 2.3% per year respectively. The corresponding figures for Pakistan in 1981 were 84.5 million and 3.0% per year respectively.
   (a) Use an algebraic method to determine when the population of Pakistan first surpasses that of China.
   (b) Which of these 3 countries would be the most populous country in 2020. Justify your answer.

7. A nuclear accident occurred in 2000. The amount, A, of radioactive substance X, remaining t years after the accident is modelled by $A = 68e^{-0.03t}$. The amount, A, of substance Y, remaining t years after the accident is modelled by $A = 40e^{-0.001t}$.
   (a) Which of these two substances is decaying at a faster rate.
   (b) Find when there will be equal amounts of X and Y.
   *(c) Find when there will be twice as much of one substance as the other.

8. In a certain ecosystem, organism A preys only on organism B. In a study conducted, the population of A (thousands) at time t months is modelled by $P = 6.98e^{0.015t}$, $t \geq 0$.
   The population of B at time t months is modelled by $P = 15.34e^{-0.02t}$, $t \geq 0$.
   (a) Find when the two populations are equal.
   (b) Comment on the validity of these models in terms of the predator-victim relationship between A and B.

9. The height H (m) of a particular species of gum tree, for a particular stage of its life, is modelled by $H = 10e^{kt}$, where t is the number of years into this stage of its life.
   (a) Find the continuous growth rate of the tree if its height when $t = 10$ is 25.4 m.
   *(b) Find the height increase in the 10th year [use the growth rate in part (a)].
   (c) If the growth rate was double that in part (a), find its height when $t = 10$ and comment on your answer.
   (d) Find the average rate of growth in the first 10 years of this stage. [use the growth rate in part(a)].

10. The weight W (tonnes) of a certain whale, during a certain stage of its life, is modelled by $W = 0.5e^{kt}$, where t is the number of years into this stage of its life.
   (a) Find the continuous growth rate of this whale if its weight when $t = 3$ is 3.5 tonnes.
   (b) Find the weight increase in the 3rd year of this stage [use the growth rate in part (a)].
   (c) Find the average rate of increase in the first 3 years of its life in this stage. [use the growth rate in part (a)].

11. The concentration C (mg/mL) of a drug compound X in the blood stream of a patient, is modelled by $C = Ae^{kt}$ where t is time in hours after the intravenous introduction of the drug (into the patients bloodstream). The drug concentration after 2 hours and after 10 hours are 1.67 mg/mL and 0.81 mg/mL respectively.
   *(a) Find the initial concentration of the drug and the continuous rate with which the drug concentration declines.
   (b) Find when the concentration drops to less than 0.5 mg/mL.

12. The number N of "mature" fish in a pond is modelled by $N = ae^{kt}$ where t is the time in months after 1st January 2008. 6 months and 12 months after 1st January 2008, the number of "mature" fish in this pond was estimated to be 7 490 and 12 480 respectively.
   (a) Find the number of "mature" fish on 1st January 2008 and the continuous rate with which the number of fish in this pond is growing.
   *(b) When the "mature" fish population reaches 15 000, 60% of the "mature" fish is harvested. Assuming that the population growth rate remains the same throughout, find the fish population 18 months after 1st January 2008.

13. The half-life of a radioactive substance decaying exponentially is 70 days. After 10 days, only 55g of the substance remained. Find the initial amount of the radioactive substance.

14. The half-life of a radioactive substance decaying exponentially is 120 years. After 5 years, only 190g of the substance remained. Find the initial amount of the radioactive substance.

15. The half-life of a radioactive substance decaying exponentially is 500 hours. Initially only 500g of the substance was manufactured. Currently there is only 20g of the substance left. How long ago was the substance manufactured?

16. 25g of a radioactive substance has decayed within the first 12 hours. Find how much would have decayed after the first hour if the half-life of this substance is 15 hours. Assume that the radioactive substance undergoes exponential decay.

02 Differentiation I

2.1 Differentiation of Polynomials

- The following results (from previous units) are quoted without proof.
  Given that a and b are constants and m and n are real numbers:
  - $\dfrac{d}{dx}(x^n) = nx^{n-1}$
  - $\dfrac{d}{dx}(ax^n \pm bx^m) = a(nx^{n-1}) \pm b(mx^{m-1})$

Example 2.1

Use Pascal's triangle or the Binomial Theorem to expand and then differentiate with respect to x: (a) $y = (1+x)^3$ (b) $y = x^3(1-x)^4$ (c) $y = \dfrac{(1+2x)^3}{x^2}$.

Solution:

(a)
$$y = (1+x)^3 = 1 + 3x + 3x^2 + x^3$$
$$\frac{dy}{dx} = 3 + 6x + 3x^2$$

(b)
$$y = x^3(1-x)^4 = x^3(1 - 4x + 6x^2 - 4x^3 + x^4) = x^3 - 4x^4 + 6x^5 - 4x^6 + x^7$$
$$\frac{dy}{dx} = 3x^2 - 16x^3 + 30x^4 - 24x^5 + 7x^6$$

(c)
$$y = \frac{(1+2x)^3}{x^2} = \frac{1 + 3(2x) + 3(2x)^2 + (2x)^3}{x^2} = x^{-2} + 6x^{-1} + 12 + 8x$$
$$\frac{dy}{dx} = -2x^{-3} - 6x^{-2} + 8$$

Notes:

- *In this chapter, we will develop rules for differentiating these functions without the need to expand them.*

2.2 Rules for Differentiation

2.2.1 The Chain Rule

Hands On Task 2.1

In this task, we will develop a rule for differentiating powers of polynomials.

1. Consider $y = (2x + 1)^2$. Let $u = 2x + 1$.
   (a) Find $\frac{dy}{dx}$ and $\frac{du}{dx}$
   (b) Verify that $y = u^2$ and hence find $\frac{dy}{du}$.
   (c) Verify that $\frac{dy}{du} \times \frac{du}{dx} = 8x + 4$.
   (d) Comment on your answers in (a) and (c).

2. Consider now, $y = (1 - x)^3$. Let $u = (1 - x)$.
   (a) Verify that $y = 1 - 3x + 3x^2 - x^3$.
   (b) Find $\frac{dy}{dx}$ and $\frac{du}{dx}$.
   (c) Find y in terms of u and hence $\frac{dy}{du}$.
   (d) Verify that $\frac{dy}{du} \times \frac{du}{dx} = -3 + 6x - 3x^2$.

3. Consider $y = \left(1 - \frac{1}{x}\right)^2$. Let $u = 1 - \frac{1}{x}$.
   (a) Verify that $y = 1 - \frac{2}{x} + \frac{1}{x^2}$.
   (b) Find $\frac{dy}{dx}$ and $\frac{du}{dx}$.
   (c) Find y in terms of u and hence $\frac{dy}{du}$.
   (d) Verify that $\frac{dy}{du} \times \frac{du}{dx} = \frac{2}{x^2} - \frac{2}{x^3}$.

4. Use your observations in Questions 1, 2 and 3 to suggest an algebraic rule that "connects" $\frac{dy}{dx}$, $\frac{dy}{du}$ and $\frac{du}{dx}$.

5. Check the accuracy of your rule on the following:
   (a) $y = (4x + 1)^3$ with $u = 4x + 1$
   (b) $y = (1 - 2x)^5$ with $u = 1 - 2x$
   (c) $y = (x^2 + 1)^4$ with $u = x^2 + 1$
   (d) $y = (1 - x^3)^4$ with $u = 1 - x^3$.

6. Given that $y = (1 + x)^4$, verify that $\frac{dy}{dx} = 4(1 + x)^3$.

7. Given that $y = (1 - 2x)^5$, verify that $\frac{dy}{dx} = 5(1 - 2x)^4 \times (-2)$.

8. Given that $y = (x^2 - 1)^6$, verify that $\frac{dy}{dx} = 6(x^2 - 1)^5 \times (2x)$.

△ Summary

- Given that $y = f(u)$ and $u = g(x)$ then:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

This is known as the *chain rule for differentiation*.

- Notice the following:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

If "cancelling" were allowed, we would get $\frac{dy}{dx}$.

Example 2.2

Use the chain rule to find $\frac{dy}{dx}$, using the suggested substitution u.

(a) $y = (1 + x)^5$ use $u = 1 + x$ (b) $y = (1 - 3x^2)^4$ use $u = 1 - 3x^2$.

Solution:

(a) $u = 1 + x \quad \Rightarrow \frac{du}{dx} = 1$

Rewrite $\quad y = u^5 \quad \Rightarrow \frac{dy}{du} = 5u^4$

Since $\quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\frac{dy}{dx} = 5u^4 \times 1 = 5(1 + x)^4$$

(b) $u = 1 - 3x^2 \quad \Rightarrow \frac{du}{dx} = -6x$

Rewrite $\quad y = u^4 \quad \Rightarrow \frac{dy}{du} = 4u^3$

Since $\quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\frac{dy}{dx} = 4u^3 \times (-6x) = -24x(1 - 3x^2)^3$$

Notes:

- *After some practice, a more mechanical technique can be used as shown.*

$$y = (1 - 3x^2)^4$$

$$\frac{dy}{dx} = 4(1 - 3x^2)^3 \times (-6x)$$

Derivative of expression within bracket

Example 2.3

Find $\frac{dy}{dx}$ for (a) $y=(x^3-1)^5$ (b) $y=\left(1+\frac{1}{x}\right)^4$.

Solution:

(a) $y=(x^3-1)^5$

$$\frac{dy}{dx}=5(x^3-1)^4\times(3x^2)=15x^2(x^3-1)^4$$

(b) $y=\left(1+\frac{1}{x}\right)^4$

$$\frac{dy}{dx}=4\left(1+\frac{1}{x}\right)^3\times\left(\frac{-1}{x^2}\right)=-\frac{4}{x^2}\left(1+\frac{1}{x}\right)^3.$$

Exercise 2.1

1. Use the rule $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$ to find:

(a) $y=(x+1)^6$, use $u=x+1$ (b) $y=(1-x)^{10}$, use $u=1-x$

(c) $y=(2x+1)^4$, use $u=2x+1$ (d) $y=(1-x/2)^8$, use $u=1-x/2$

(e) $y=\left(1+\frac{x}{3}\right)^{\frac{3}{2}}$, use $u=1+\frac{x}{3}$ (f) $y=\left(\frac{x}{5}-1\right)^{\frac{5}{2}}$, use $u=\frac{x}{5}-1$

(g) $y=(1+2x^2)^4$, use $u=1+2x^2$ (h) $y=(x^3-1)^5$, use $u=x^3-1$

(i) $y=(5-x^5)^{5/2}$, use $u=5-x^5$ (j) $y=(x^2-4)^{1/2}$, use $u=x^2-4$

2. Use the chain rule to differentiate y with respect to x:

(a) $y=(1+3x^2)^4$ (b) $y=(x^2/2-2)^3$ (c) $y=(6+x^2)^{3/2}$

(d) $y=(3+x^3)^{5/3}$ (e) $y=(1-1/x^2)^5$ (f) $y=(1/x^3+1)^4$

(g) $y=(1+\sqrt{x})^4$ (h) $y=(2+x^{1/3})^3$ (i) $y=(x^{1/3}-1)^{3/2}$

3. Use the chain rule to find:

(a) $\frac{d}{dx}(1+x+x^2)^3$ (b) $\frac{d}{dx}(1-2x+x^3)^4$ (c) $\frac{d}{dx}(1+\frac{1}{x}+x)^3$ (d) $\frac{d}{dx}(1+\sqrt{x}+x)^4$

4. Use the chain rule to find $f'(x)$:

(a) $f(x)=\frac{1}{1+x}$ (b) $f(x)=\frac{1}{(1-x)^2}$ (c) $f(x)=\frac{1}{(2+x^2)}$ (d) $f(x)=\frac{2}{(1-x^2)^2}$

5. Find y':

(a) $y=\frac{4}{1-\sqrt{x}}$ (b) $y=\frac{3}{\sqrt{1+x}}$ (c) $y=\frac{-1}{(1+\sqrt{x})^2}$ (d) $y=\frac{1}{1+x+x^2}$

*6. Differentiate with respect to t:

(a) $v = \sqrt{1+t^2}$ (b) $v = \dfrac{1}{\sqrt{1+t^2}}$ (c) $v = 1 + t^2 + \dfrac{1}{1+t}$ (d) $v = \left(1+\dfrac{1}{1+t}\right)^3$

7. Given that $f(t) = (1 + t^2)^2$, find $f'(t)$. Hence, find $f'(1)$.

8. Given that $f(t) = \dfrac{1}{16-t^2}$, find $f'(t)$. Hence, find $f'(1)$.

9. Given that $f(t) = (1 - 2t^3)^4$, find t such that $f'(t) = 0$.

10. Given that $f(t) = (t/4 - 1)^4 - 1$, find t such that $f'(t) = 0$.

2.2.2 The Product Rule

Hands On Task 2.2

In this task, we will explore a rule for differentiating products of polynomials.

1. Consider $y = (2x + 1)(3x + 1)$. Let $u = 2x + 1$ and $v = 3x + 1$.
   (a) Expand $(2x + 1)(3x + 1)$. Hence, find $\dfrac{dy}{dx}$. (b) Find $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$.
   (c) Find $\left(\dfrac{du}{dx}\times v\right) + \left(u\times\dfrac{dv}{dx}\right)$ and comment on your answer.

2. Consider $y = (x^2 - 1)(x^3 - 1)$. Let $u = x^2 - 1$ and $v = x^3 - 1$.
   (a) Expand $(x^2 - 1)(x^3 - 1)$. Hence, find $\dfrac{dy}{dx}$. (b) Find $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$.
   (c) Find $\left(\dfrac{du}{dx}\times v\right) + \left(u\times\dfrac{dv}{dx}\right)$ and comment on your answer.

3. Consider $y = (1 + 2x^2)(1 - 2x^3)$. Let $u = 1 + 2x^2$ and $v = 1 - 2x^3$.
   (a) Find $\dfrac{dy}{dx}$. (b) Find $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$.
   (c) Find $\left(\dfrac{du}{dx}\times v\right) + \left(u\times\dfrac{dv}{dx}\right)$ and comment on your answer.

4. Given that $y = u \times v$, where u and v are expressions in x, use your observations in Questions 1, 2 and 3 above, to describe using appropriate symbols, a rule for $\dfrac{dy}{dx}$.

△ Summary

- Given that $y = u \times v$, where $u = f(x)$ and $v = g(x)$,

$$\frac{dy}{dx} = \left(\frac{du}{dx} \times v\right) + \left(u \times \frac{dv}{dx}\right)$$

This is called the ***product rule*** for differentiation.

- Note the pattern embedded in this rule:

$$y = u \times v$$

$$\frac{dy}{dx} = \left(\frac{du}{dx} \times v\right) + \left(u \times \frac{dv}{dx}\right)$$

↑ differentiate u, leave v

↑ ADD

↑ differentiate v, leave u

Example 2.4

Use the product rule to find $\frac{dy}{dx}$. (a) $y = (2x + 1)(5x + 1)$ (b) $y = (\sqrt{x} + 1)(x^2 - 1)$.

Solution:

(a)

$$y = (2x + 1)(5x + 1)$$

$$\frac{dy}{dx} = \left[\frac{d}{dx}(2x+1)\right](5x + 1) + (2x + 1)\left[\frac{d}{dx}(5x+1)\right]$$

$$= 2(5x + 1) + (2x + 1)\,5$$

$$= 10x + 2 + 10x + 5$$

$$= 20x + 7$$

(b)

$$y = (\sqrt{x} + 1)(x^2 - 1) = (x^{\frac{1}{2}} + 1)(x^2 - 1)$$

$$\frac{dy}{dx} = \left[\frac{d}{dx}(x^{\frac{1}{2}} + 1)\right](x^2 - 1) + (x^{\frac{1}{2}} + 1)\left[\frac{d}{dx}(x^2 - 1)\right]$$

$$= \frac{1}{2}x^{-\frac{1}{2}}(x^2 - 1) + 2x(x^{\frac{1}{2}} + 1)$$

$$= \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{2}x^{-\frac{1}{2}} + 2x^{\frac{3}{2}} + 2x$$

$$= \frac{5}{2}x^{\frac{3}{2}} + 2x - \frac{1}{2}x^{-\frac{1}{2}}$$

Example 2.5

Use the chain rule and product rule to find $\frac{dy}{dx}$.

(a) $y = (2x+1)^3(x^2-1)$ (b) $y = (x+1)(x^2-1)^{1/2}$

Solution:

(a)

$$y = (2x+1)^3(x^2-1)$$
$$\frac{dy}{dx} = \left[\frac{d}{dx}(2x+1)^3\right](x^2-1) + (2x+1)^3\left[\frac{d}{dx}(x^2-1)\right]$$
$$= [3(2x+1)^2(2)](x^2-1) + (2x+1)^3(2x)$$
$$= 6(2x+1)^2(x^2-1) + 2x(2x+1)^3 \quad \longleftarrow \text{Answer in unfactored form}$$
$$= 2(2x+1)^2[3(x^2-1) + x(2x+1)]$$
$$= 2(2x+1)^2[5x^2+x-3] \quad \longleftarrow \text{Answer in factored form}$$

(b)

$$y = (x+1)(x^2-1)^{1/2}$$
$$\frac{dy}{dx} = \left[\frac{d}{dx}(x+1)\right](x^2-1)^{1/2} + (x+1)\left[\frac{d}{dx}(x^2-1)^{1/2}\right]$$
$$= (x^2-1)^{1/2} + (x+1)[\frac{1}{2}(x^2-1)^{-1/2}(2x)]$$
$$= (x^2-1)^{1/2} + x(x+1)(x^2-1)^{-1/2} \quad \longleftarrow \text{Answer in unfactored form}$$
$$= (x^2-1)^{-1/2}[(x^2-1) + x(x+1)]$$
$$= (x^2-1)^{-1/2}[2x^2+x-1] \quad \longleftarrow \text{Answer in factored form}$$

Exercise 2.2

1. Use the product rule to find dy/dx. *You may leave your answers in an unsimplified form*

(a) $y = (x-2)(3x+1)$ (b) $y = (1-x)(2x+3)$

(c) $y = (1+\frac{x}{2})(4-\frac{x}{3})$ (d) $y = (1+x+x^2)(4-x^2)$

(e) $y = (1+\sqrt{x})(1-x^2)$ (*f*) $y = (1+\frac{1}{x})(1+x-x^2)$

(g) $y = (2+\frac{1}{\sqrt{x}})(x^3+1)$ (h) $y = (x+\frac{1}{x})(x^2-4)$

2. Use the product rule to find $\frac{d}{dx}f(x)$.

(a) $\frac{d}{dx}(x+1)^2(x+2)$ (b) $\frac{d}{dx}(1-x)(x+3)^3$

*(c) $\frac{d}{dx}(x^2+x+1)(2x+1)^3$ *(d) $\frac{d}{dx}(x^2+x+1)(1-x-x^2)^3$

3. Find y'.

(a) $y = (2x + 1)^{3/2}(x + 2)$
(b) $y = (1 - x)(4x + 3)^{3/2}$
(c) $y = (1 + 2x)^{1/2}(x - 1)$
(d) $y = (1 + 3x)\sqrt{3 - 2x}$

4. Differentiate with respect to t.

(a) $f(t) = (t + 1)^2(t - 2)^3$
(b) $f(t) = (2t + 1)^2(t + 2)^2$
(c) $f(t) = (1 - \frac{t}{2})^2(1 + 2t)^3$
(d) $f(t) = (1 + 2t)^{3/2}(1 - t)^2$
*(e) $v(t) = (1 + 4t)^{3/2}(3 + 2t)^{5/2}$
*(f) $v(t) = (1 + 3t)^{1/3}(1 + 2t)^{3/2}$
*(g) $P(t) = (1 + 3t)^{4/3}(1 + 2t)^{1/2}$
*(h) $P(t) = (1 + 5t)^{1/5}(1 + 6t)^{1/3}$

5. Differentiate with respect to t.

(a) $v = (t^2 + 1)^2(1 - t^3)^2$
(b) $y = (t^2 - 1)^3(1 - \frac{t^4}{2})^2$

6. Use the product rule to find dy/dt.

(a) $y = (t + 1)(t + 2)^{-1}$
(b) $y = \frac{t+3}{1-t}$
(c) $y = \frac{(t+1)^2}{2+t}$
(d) $y = \frac{t-1}{(1+t)^2}$

7. Use the product rule to find y'.

(a) $y = (x + 1)(x - 1)^{-1/2}$
(b) $y = (x^2 + 1)^{-3/2}(x - 1)$
(c) $y = \frac{t^2}{\sqrt{1+t}}$
(d) $y = \frac{2t}{(1-t^2)^{\frac{3}{2}}}$

8. Given that $f(x) = (x + 1)^2(1 - x)$. Find $f'(x)$. Hence, find $f'(0)$.

9. Given that $f(x) = (1 - x^2)(1 + 2x)^3$. Find $f'(x)$. Hence, find $f'(1)$.

10. Given that $v(t) = (1 + t)(1 - t)^2$, find $v'(t)$. Hence, find t such that $v'(t) = 0$.

*11. Given that $P(t) = (1 - \frac{t}{2})^2(1 + 2t^2)$. Find $P'(t)$. Hence, find t such that $P'(t) = 0$.

*12. Given that $s(t) = (1 + 2t)^{3/2}(1 - t^2)$. Find $s'(t)$. Hence; find t such that $s'(t) = 0$.

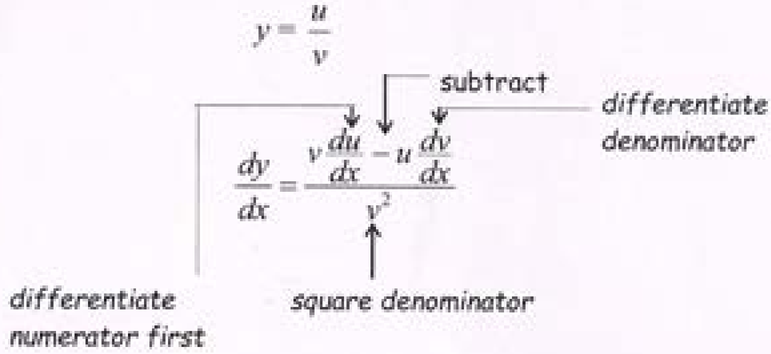
2.2.3 The Quotient Rule

- Given that $y = \dfrac{u}{v}$, where $u = f(x)$ and $v = g(x)$,

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

This is known as the *quotient rule* for differentiation.

- Note the pattern embedded in this rule.



- Remember that quotients can also be differentiated using the product rule.

Example 2.6

Use the quotient rule to find $\dfrac{dy}{dt}$: (a) $y = \dfrac{1+t}{1-t}$ (b) $y = \dfrac{1-t}{1-2t^2}$

Solution:

(a) $y = \dfrac{1+t}{1-t} \Rightarrow \dfrac{dy}{dt} = \dfrac{(1-t)(1)-(1+t)(-1)}{(1-t)^2}$

$$= \frac{(1-t)+(1+t)}{(1-t)^2}$$

$$= \frac{2}{(1-t)^2}$$

Reworking (a) using the product rule:

$$y = (1+t)(1-t)^{-1}$$

$$\frac{dy}{dt} = (1)(1-t)^{-1} + (1+t)[(-1)(1-t)^{-2}(-1)]$$

$$= (1-t)^{-1} + (1+t)(1-t)^{-2}$$

$$= (1-t)^{-2}[(1-t)+(1+t)] = 2/(1-t)^2$$

(b) $y = \dfrac{1-t}{1-2t^2} \Rightarrow \dfrac{dy}{dt} = \dfrac{(1-2t^2)(-1)-(1-t)(-4t)}{(1-2t^2)^2}$

$$= \frac{-(1-2t^2)+4t(1-t)}{(1-2t^2)^2}$$

$$= \frac{-1+2t^2+4t-4t^2}{(1-2t^2)^2}$$

$$= \frac{-2t^2+4t-1}{(1-2t^2)^2}$$

Example 2.7

Use the quotient rule and the chain rule to find $\frac{dy}{dt}$: (a) $y=\frac{(1-t)^2}{(1+t)}$ (b) $y=\frac{(1-t)}{(1+2t)^2}$

Solution:

(a) $y=\frac{(1-t)^2}{(1+t)} \Rightarrow \frac{dy}{dt}=\frac{(1+t)[2(1-t)(-1)]-(1-t)^2(1)}{(1+t)^2}$

$$=\frac{-2(1+t)(1-t)-(1-t)^2}{(1+t)^2}$$

← Answer in unfactored form

$$=\frac{(1-t)[-2(1+t)-(1-t)]}{(1+t)^2}$$

$$=\frac{(1-t)[-3-t]}{(1+t)^2}=\frac{(t-1)[t+3]}{(1+t)^2}$$

← Answer in factored form

(b) $y=\frac{(1-t)}{(1+2t)^2} \Rightarrow \frac{dy}{dt}=\frac{(1+2t)^2(-1)-(1-t)2(1+2t)^1(2)}{(1+2t)^4}$

$$=\frac{-(1+2t)^2-4(1-t)(1+2t)}{(1+2t)^4}$$

← Answer in unfactored form

$$=\frac{(1+2t)[-(1+2t)-4(1-t)]}{(1+2t)^4}$$

$$=\frac{-5+2t}{(1+2t)^3}$$

← Answer in factored form

Exercise 2.3

1. Use the quotient rule to differentiate each of the following with respect to x:
   (a) $1/x^2$ (b) $1/(4x^3)$ (c) $1/(x+1)$ (d) $-1/(x+1)^2$
   (e) $1/(x^2+1)$ (f) $1/(x+1)^{3/2}$ (g) $-1/(x-1)^{1/2}$ (h) $1/(1-x)^{3/2}$

2. Use the quotient rule to find $f'(x)$.
   (a) $f(x)=\frac{x}{1-x}$ (b) $f(x)=\frac{-2x}{1+3x}$ (c) $f(x)=\frac{1+x}{1-6x}$ (d) $f(x)=\frac{1+2x}{1-3x}$
   (e) $f(x)=\frac{x^2}{1+x}$ (f) $f(x)=\frac{-2x}{1-x^2}$ (g) $f(x)=\frac{-x^2}{1+x^2}$ (h) $f(x)=\frac{1+2x^2}{1+x^2}$

3. Use the quotient rule to find y'. *You may leave your answers in an unfactored form.*
   (a) $y=\frac{x}{(1+x)^2}$ (b) $y=\frac{-2x}{(1-x)^2}$ (c) $y=\frac{x^2}{(1-2x)^2}$ (d) $y=\frac{-4x^3}{(2+x)^2}$
   (e) $y=\frac{2x}{(1+x^2)^2}$ (f) $y=\frac{-x^2}{(1-x^2)^2}$ (g) $y=\frac{1+x}{(1-x)^2}$ (h) $y=\frac{(1+x)^2}{1-x}$
   *(i) $y=\frac{(4-x)^2}{(1+2x)^2}$ *(j) $y=\frac{(3+2x)^2}{(4-5x)^2}$ *(k) $y=\frac{1+x^2}{(1-3x)^2}$ *(l) $y=\frac{(1-x^2)^2}{(1+x^2)^2}$

4. Differentiate with respect to t. *You may leave your answers in an unfactored form.*

(a) $\dfrac{1}{(4t-3)^{3/2}}$ (b) $\dfrac{-2}{(1+2t)^{1/2}}$ (c) $\dfrac{4}{(1-\sqrt{t})}$ (d) $\dfrac{6}{(1+\sqrt{t})^{3/2}}$

(e) $\dfrac{2t}{(1+t)^{3/2}}$ (f) $\dfrac{-4t}{(1-2t)^{1/2}}$ (g) $\dfrac{(1+2t)^{3/2}}{t}$ (h) $\dfrac{(3-4t)^{1/2}}{2t}$

5. Differentiate with respect to x. *You may leave your answers in an unfactored form.*

(a) $\dfrac{1+x}{(1-x)^{3/2}}$ (b) $\dfrac{1-x}{(1+2x)^{1/2}}$ (c) $\dfrac{(1+2x)^{3/2}}{x}$ (d) $\dfrac{(3-4x)^{1/2}}{1+2x}$

(e) $\dfrac{-2x}{(1+x^2)^{3/2}}$ (f) $\dfrac{x^2}{(1-x^2)^{1/2}}$ (g) $\dfrac{1+x}{(1-x^2)^{3/2}}$ (h) $\dfrac{(1-x^2)^{1/2}}{(1+x^2)}$

*6. Find $dy/d\theta$. *You may leave your answers in an unfactored form.*

(a) $y=\sqrt{\dfrac{1+\theta}{1-\theta}}$ (b) $y=\sqrt{\dfrac{1-\theta^2}{1+\theta^2}}$

7. Given that $f(x)=\dfrac{x}{1+x^2}$, find: (a) $f'(0)$ (b) x such that $f'(x)=0$.

8. Given that $f(x)=\dfrac{1+x}{4-9x^2}$, find: (a) $f'(0)$ (b) x such that $f'(x)=0$.

*9. Given that $v(t)=\dfrac{t^2}{(1+2t)^2}$, find: (a) $v'(0)$ (b) t such that $v'(t)=0$

*10. Given that $v(t)=\dfrac{t}{(1+t)^{3/2}}$, find: (a) $v'(0)$ (b) t such that $v'(t)=0$

2.3 Higher Derivatives

- $\dfrac{dy}{dx}$ or y' or $f'(x)$ is called the *first derivative* of y or $f(x)$ with respect to x.
- The derivative of the first derivative is called the *second derivative* of y or $f(x)$ with respect to x. This is denoted $\dfrac{d^2y}{dx^2}$ or y'' or $\dfrac{d^2}{dx^2}f(x)$ or $f''(x)$.
- The derivative of the second derivative is called the *third derivative* of y or $f(x)$ with respect to x. This is denoted $\dfrac{d^3y}{dx^3}$ or $\dfrac{d^3}{dx^3}f(x)$ or $f'''(x)$ or y'''.
- In general, the *nth derivative* of y or $f(x)$ with respect to x, is denoted

  $\dfrac{d^ny}{dx^n}$ or $y^{(n)}$ or $\dfrac{d^n}{dx^n}f(x)$ or $f^{(n)}(x)$.

  [(n) is the Roman numeral equivalent of the Arabic numeral denoted by n.]

Example 2.8

Given that $f(x) = (1 + x^2)^3$, find $f'(x)$ and $f''(x)$.

Solution:

$$f(x) = (1 + x^2)^3$$

Hence,

$$f'(x) = 3(1 + x^2)^2 (2x)$$
$$= 6x(1 + x^2)^2$$

Therefore

$$f''(x) = [6](1 + x^2)^2 + 6x[2(1 + x^2)(2x)]$$

← Answer in unfactored form

$$= 6(1 + x^2)^2 + 24x^2(1 + x^2)$$
$$= 6(1 + x^2)[(1 + x^2) + 4x^2]$$
$$= 6(1 + x^2)[1 + 5x^2]$$

← Answer in factored form

Exercise 2.4

1. Given $\dfrac{dy}{dx}$, find $\dfrac{d^2y}{dx^2}$.

   (a) $\dfrac{dy}{dx} = 6x^2$ (b) $\dfrac{dy}{dx} = \dfrac{1}{4x^2}$ (c) $\dfrac{dy}{dx} = (x+1)^3$ (d) $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x+1}}$

2. Given $f'(x)$, find $f''(x)$. *You may leave your answers in an unfactored form.*

   (a) $f'(x) = x(1 + x)^3$ (b) $f'(x) = x^4(1 + \sqrt{x})^2$

   (c) $f'(x) = (1 + x)^{3/2}(1 + 3x)$ *(d) $f'(x) = (1 - x^2)^{3/2}(1 + 4x)$

3. Given $y = f(x)$, find y''.

   (a) $y = (1 + 2x)^4$ (b) $y = (2 - 15x)^{3/2}$ (c) $y = (1 - 6x)^{3/2}$ (d) $y = 1/\sqrt{(1 - 4x)}$

4. Given $v = f(t)$, find the second derivative of v with respect to t.

   (a) $v = (1 + t^2)^2$ (b) $v = (1 - t^2)^3$

5. For each of the following, find $\dfrac{d^3y}{dx^3}$ and hence suggest an expression for $\dfrac{d^5y}{dx^5}$:

   (a) $y = x^{10}$ (b) $y = x^{-1/2}$

6. Given that $f(x) = (5 + 2x)^4$, find : (a) $f''(0)$ (b) x such that $f''(x) = 0$.

7. For each of the following, find the value(s) of $f''(x)$ corresponding to the value(s) of x that makes $f'(x) = 0$.

   (a) $f(x) = (x - 1)^6$ (b) $f(x) = x^3 - 12x$ (c) $f(x) = 2x^3 - 9x^2 + 12x$

03 Differentiation II

3.1 Differentiating Exponential Functions

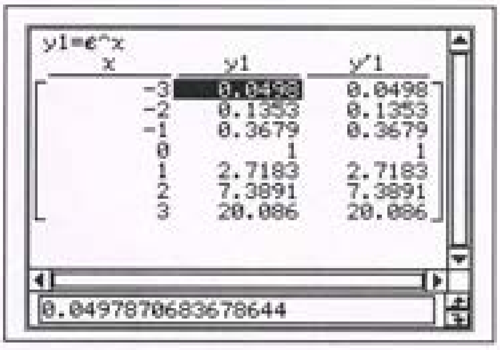
3.1.1 Differentiating e^{mx}

Hands On Task 3.1

In this task, we will explore a rule for differentiating expressions of the form e^{mx}.

1. Turn on the "derivative" facility in your CAS calculator. Tabulate on your CAS calculator, $y = e^x$ for $-3 \le x \le 3$ for unit increments for x. With the "derivative" facility on, the equivalent values of $\frac{dy}{dx}$ are simultaneously shown.

   Comment on the values of y and $\frac{dy}{dx}$.



2. Repeat question 1 for:
   (a) $y = e^{2x}$ (b) $y = e^{3x}$ (c) $y = e^{-x}$ (d) $y = e^{0.5x}$

3. Predict the value of $\frac{dy}{dx}$ for each of the following at the point where $x = 1$:
   (a) $y = e^{5x}$ (b) $y = e^{-2x}$ (c) $y = e^{-0.05x}$ (d) $y = 2e^{4x}$

4. Use your observations above to suggest a rule for differentiating e^{mx} with respect to x.

△ Summary

- Given $y = e^{mx}$, $\frac{dy}{dx} = me^{mx}$.
- Note the pattern:

$$y = e^{mx}$$

$$\frac{dy}{dx} = me^{mx}$$

Note: • *The proof for the derivative of $y = e^x$ is given in Example 4.2.*

Example 3.1

Find $\frac{dy}{dx}$ for: (a) $y = e^{0.01x}$ (b) $y = \frac{1}{e^{3x}}$ (c) $y = x^2 e^x$ (d) $y = \frac{x}{e^x}$.

Solution:

(a) $y = e^{0.01x} \Rightarrow \frac{dy}{dx} = 0.01e^{0.01x}$

(b) $y = \frac{1}{e^{3x}} = e^{-3x} \Rightarrow \frac{dy}{dx} = -3e^{-3x}$

(c) $y = x^2 e^x \Rightarrow \frac{dy}{dx} = 2x \times e^x + x^2 \times e^x = xe^x(2 + x)$

(d) $y = \frac{x}{e^x} = xe^{-x} \Rightarrow \frac{dy}{dx} = e^{-x} + x(-e^{-x}) = e^{-x}(1 - x)$.

Exercise 3.1

1. Find $\frac{dy}{dx}$:

(a) $y = e^{0.05x}$ (b) $y = e^{1.2x}$ (c) $y = e^{-0.05x}$

(d) $y = e^{-1.1x}$ (e) $y = 1/e^{0.02x}$ (f) $y = 1/e^{10x}$

(g) $y = 50\,e^{0.01x}$ (h) $y = 3/e^{0.03x}$ (i) $y = \frac{5}{4e^{0.09x}}$

2. Find y'.

(a) $y = x^2 + 3e^x$ (b) $y = 1/x + 4e^{x/4}$ (c) $y = (e^x + 1)^2$

(d) $y = e^{4x} + e^{-4x}$ (e) $y = (x + 1)^4 + e^{5x}/5$ (f) $y = (1 - x^2)^3 + 4e^{-3x}$

(g) $y = \frac{1}{4e^{4x}} + \frac{1}{(x+1)^2}$ (h) $y = \frac{1}{2e^{-2x}} + \frac{1}{(1-x)^2}$ (i) $y = \frac{e^x + 1}{e^{2x}}$

3. Find $\frac{dy}{dt}$. *You may leave your answers in an unfactored form.*

(a) $y = t^2 e^{-t}$ (b) $y = 2t^2 e^{t/2}$ (c) $y = (1 + t^2)e^t$

(d) $y = (1 - t^3)e^t$ (e) $y = (1 - t)^2 e^{-t}$ (f) $y = (1 + t^2)^2 e^{2t}$

(g) $y = t^2/e^t$ (h) $y = e^{2t}/t^2$ (i) $y = e^{-t}/(1 + t)^2$

4. For each of the following, find: (i) $f'(2)$ (ii) x such that $f'(x) = 0$:

(a) $f(x) = 2x^3 e^x$ (b) $f(x) = x^3/e^{x/2}$

*5. For each of the following, find the value(s) of $f''(x)$ that correspond to $f'(x) = 0$:

(a) $f(x) = x^2 e^x$ (b) $f(x) = (1 + x)e^x$.

3.1.2 Differentiating $e^{f(x)}$

- Consider $y = e^{f(x)}$.

  Let $u = f(x)$. Hence, $y = e^{u}$.
  Using the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= e^{u} \times f'(x)$$
$$= f'(x)\, e^{f(x)}.$$

- Note the pattern.

$$y = e^{f(x)}$$

differentiate

$$\frac{dy}{dx} = f'(x)\, e^{f(x)}$$

Example 3.2

Find $f'(x)$ if (a) $f(x) = e^{x^2+1}$ (b) $f(x) = xe^{x^2}$.

Solution:

(a) $f(x) = e^{x^2+1} \Rightarrow f'(x) = 2x \times e^{x^2+1} = 2xe^{x^2+1}$

(b) $f(x) = xe^{x^2} \Rightarrow f'(x) = 1 \times e^{x^2} + x \times (2x)\, e^{x^2}$

$$= e^{x^2} + 2x^2 e^{x^2}$$
$$= e^{x^2}(1 + 2x^2)$$

Exercise 3.2

1. Find y'.

   (a) $y = e^{-x^2}$ (b) $y = e^{-x^2+2x}$ (c) $y = 4e^{2x^2-x}$ (d) $y = \frac{1}{3}e^{x^3}$

   (e) $y = 2e^{\sqrt{x}}$ (f) $y = e^{1/x}$ (g) $y = 1/e^{x+x^2}$ (h) $4/e^{\sqrt{x}}$

2. Find dv/dt.

   (a) $v = t^2 e^{t^2}$ (b) $v = (1+t)e^{t^2+1}$ (c) $v = \dfrac{1+t}{2e^{t^2}}$ (d) $v = \dfrac{e^{t^2}}{1+t}$

*3. For each of the following find the value(s) of $P'(t)$ that correspond to $P''(t) = 0$:

   (a) $P(t) = -e^{-t^2}$ (b) $P(t) = te^{-t^2}$.

04 Differentiation III

4.1 First Principles

- In this section, we will review the concept of the derivative using first principles.
- The derivative of $f(x)$, denoted $f'(x)$ or $\frac{d}{dx}f(x)$, is defined as:

$$f'(x) = \frac{d}{dx}f(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$$

Example 4.1

Use the definition of the derivative to find the derivative of: (a) $5x^2$ (b) $\frac{1}{2x^2}$.

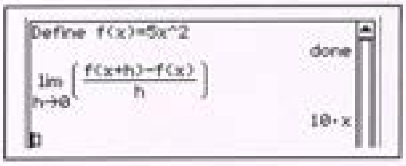
Solution:

(a) Let $f(x) = 5x^2$.

Hence, $f'(x) = \lim_{h\to 0}\left[\frac{5(x+h)^2-5x^2}{h}\right]$

$$= \lim_{h\to 0}\left[\frac{5(x^2+2hx+h^2)-5x^2}{h}\right]$$

$$= \lim_{h\to 0}(10x+5h) = 10x$$



(b) Let $f(x) = \frac{1}{2x^2}$

Hence, $f'(x) = \lim_{h\to 0}\left[\frac{\frac{1}{2(x+h)^2}-\frac{1}{2x^2}}{h}\right]$

$$= \lim_{h\to 0}\left[\frac{x^2-(x+h)^2}{2x^2(x+h)^2h}\right]$$

$$= \lim_{h\to 0}\left[\frac{-h(2x+h)}{2hx^2(x+h)^2}\right] = \lim_{h\to 0}\left[\frac{-(2x+h)}{2x^2(x+h)^2}\right]$$

$$= -\frac{2x}{2x^4} = -\frac{1}{x^3}$$

Example 4.2

Without the use of a CAS calculator, find the derivative of $y = e^x$.

Solution:

$$\frac{dy}{dx} = \lim_{h \to 0}\left[\frac{e^{x+h} - e^x}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{e^x(e^h - 1)}{h}\right]$$

$$= \lim_{h \to 0}\left[e^x \times \frac{(e^h - 1)}{h}\right]$$

$$= e^x \times 1 = e^x$$

From Chapter 1:

$$\lim_{h \to 0}\left[\frac{(e^h - 1)}{h}\right] = 1$$

Example 4.3

Identify the function that is being differentiated in $\lim_{h \to 0}\left[\frac{-2(x+h)^5 + 2x^5}{h}\right]$.

Hence, evaluate this limit.

Solution:

Rewrite $\lim_{h \to 0}\left[\frac{-2(x+h)^5 + 2x^5}{h}\right]$ as $\lim_{h \to 0}\left[\frac{-2(x+h)^5 - (-2x^5)}{h}\right]$

Compare $\lim_{h \to 0}\left[\frac{-2(x+h)^5 - (-2x^5)}{h}\right]$ with $\lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$.

Clearly, $f(x) = -2x^5$.

Hence, the function that is being differentiated is $f(x) = -2x^5$.

Therefore, $$\lim_{h \to 0}\left[\frac{-2(x+h)^5 - (-2x^5)}{h}\right] = \frac{d}{dx}(-2x^5)$$

$$= -10x^4$$

Example 4.4

Without the use of a CAS/graphic calculator, find the exact value of $\lim_{h\to 0}\left[\dfrac{\sqrt{\left(\frac{1}{4}+h\right)}-\frac{1}{2}}{h}\right]$.

Solution:

Rewrite $\lim_{h\to 0}\left[\dfrac{\sqrt{\left(\frac{1}{4}+h\right)}-\frac{1}{2}}{h}\right]$ as $\lim_{h\to 0}\left[\dfrac{\sqrt{\left(\frac{1}{4}+h\right)}-\sqrt{\frac{1}{4}}}{h}\right]$

Compare $\lim_{h\to 0}\left[\dfrac{\sqrt{\left(\frac{1}{4}+h\right)}-\sqrt{\frac{1}{4}}}{h}\right]$ with $\lim_{h\to 0}\left[\dfrac{f(x+h)-f(x)}{h}\right]$.

$$\text{Clearly, } \lim_{h\to 0}\left[\frac{\sqrt{\left(\frac{1}{4}+h\right)}-\sqrt{\frac{1}{4}}}{h}\right]=\frac{d}{dx}[\sqrt{x}]\bigg|_{x=\frac{1}{4}}$$

$$=\frac{1}{2\sqrt{x}}\bigg|_{x=\frac{1}{4}}=1$$

Exercise 4.1

1. Without the use of a calculator, use first principles to differentiate the following:
   (a) -5 (b) $x/2$ (c) x^3 (d) $3x^2-2x$ (e) e^{2x}

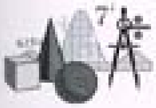
2. Evaluate each of the following limits by identifying the function being differentiated.
   (a) $\lim_{h\to 0}\left[\dfrac{3(x+h)^4-3x^4}{h}\right]$ (b) $\lim_{h\to 0}\left[\dfrac{2(y+h)^3-2y^3}{3h}\right]$ (c) $\lim_{h\to 0}\left[\dfrac{\frac{4}{3(x+h)^2}-\frac{4}{3x^2}}{h}\right]$
   (d) $\lim_{h\to 0}\left[\dfrac{\sqrt{2(t+h)}-\sqrt{2t}}{h}\right]$ (e) $\lim_{h\to 0}\left[\dfrac{e^{2(x+h)}-e^{2x}}{h}\right]$ (f) $\lim_{h\to 0}\left[\dfrac{\frac{1}{e^{x+h}}-\frac{1}{e^x}}{h}\right]$

3. Find the exact values of each of the following:
   (a) $\lim_{h\to 0}\left[\dfrac{(h+2)^2-4}{h}\right]$ (b) $\lim_{h\to 0}\left[\dfrac{(h-2)^3+8}{h}\right]$ (c) $\lim_{h\to 0}\left[\dfrac{\frac{1}{(1+h)^2}-1}{h}\right]$
   (d) $\lim_{x\to 0}\left[\dfrac{\sqrt{25+x}-5}{x}\right]$ (e) $\lim_{h\to 0}\left[\dfrac{e^{h-3}-e^{-3}}{h}\right]$ (f) $\lim_{h\to 0}\left[\dfrac{(2+h)e^{2+h}-2e^2}{h}\right]$

4.2 Differentiating Trigonometric Functions

- A list of important trigonometric limits is given below (x must be in radians):

> - $\lim_{x \to 0} \cos(x) = 1$
> - $\lim_{x \to 0} \dfrac{1-\cos(x)}{x} = 0$
> - $\lim_{x \to 0} \dfrac{\sin(x)}{x} = 1$ [This implies that for small x, $\sin(x) \approx x$]
> - $\lim_{x \to 0} \dfrac{\tan(x)}{x} = 1$ [This implies that for small x, $\tan(x) \approx x$]



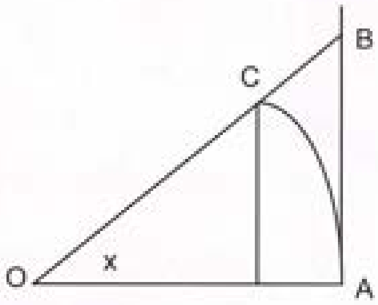
Hands On Task 4.1

In this task, we will explore the development of some of the trigonometric limits listed above.

1. Use your CAS calculator to verify the following limits (x is in ***radians***):

   (a) $\lim_{x \to 0} \cos(x) = 1$ (b) $\lim_{x \to 0} \dfrac{1-\cos(x)}{x} = 0$ (c) $\lim_{x \to 0} \dfrac{\sin(x)}{x} = 1$ (d) $\lim_{x \to 0} \dfrac{\tan(x)}{x} = 1$.

2. The result for $\lim_{x \to 0} \dfrac{\sin(x)}{x}$ can also be determined using geometry. The accompanying diagram shows a sector of a circle OAC, with OA = OC being the radius, and a central angle of size x radians.
   Triangle OAB is a right angled triangle.



   Clearly:
   Area of triangle OAC < Area of sector OAC < Area of triangle OAB.

   (a) Use this result to show that $\sin(x) < x < \tan(x)$.

   (b) Hence, show that $1 < \dfrac{x}{\sin(x)} < \dfrac{1}{\cos(x)}$ and $\cos(x) < \dfrac{\sin(x)}{x} < 1$.

   (c) Hence, find $\lim_{x \to 0} \dfrac{\sin(x)}{x}$.

3. The result for $\lim_{x \to 0} \dfrac{1-\cos(x)}{x}$ can also be obtained algebraically.

   Rewrite $\lim_{x \to 0} \dfrac{1-\cos(x)}{x}$ as $\lim_{x \to 0} \left[\dfrac{1}{x} - \dfrac{\cos(x)}{x}\right]$. Hence find $\lim_{x \to 0} \dfrac{1-\cos(x)}{x}$.

4.2.1 Derivative of sin (x)

- To find the derivative of sin (x), ***where x is in radians***, we adopt the following procedure.

- Let $f(x) \equiv \sin(x)$.

  Using first principles:

$$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\left[\frac{\sin(x+h)-\sin(x)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\sin(x)[\cos(h)-1]+\cos(x)\sin(h)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{\sin(x)[\cos(h)-1]}{h}+\frac{\cos(x)\sin(h)}{h}\right]$$

$$= \lim_{h\to 0}\left[\sin(x)\left[\frac{\cos(h)-1}{h}\right]+\cos(x)\left[\frac{\sin(h)}{h}\right]\right]$$

$$= \cos(x)$$

Using the results:

$$\lim_{h\to 0}\left[\frac{1-\cos(h)}{h}\right]=0$$

$$\lim_{h\to 0}\left[\frac{\sin(h)}{h}\right]=1$$

4.2.2 Derivatives of basic trigonometric functions

- Listed below are some commonly used results involving trigonometric functions:

- $\frac{d}{dx}\sin(x)=\cos(x)$
- $\frac{d}{dx}\cos(x)=-\sin(x)$
- $\frac{d}{dx}\tan(x)=\frac{1}{\cos^2(x)}=\sec^2(x)$
- $\frac{d}{dx}\cot(x)=\frac{-1}{\sin^2(x)}=-\operatorname{cosec}^2(x)$

Example 4.5

Without the use of a CAS calculator, find the exact value of $\lim_{h\to 0}\left[\dfrac{\sin\left(\frac{\pi}{3}+h\right)-\sin\left(\frac{\pi}{3}\right)}{h}\right]$.

Solution:

$$\lim_{h\to 0}\left[\frac{\sin\left(\frac{\pi}{3}+h\right)-\sin\left(\frac{\pi}{3}\right)}{h}\right] = \frac{d}{dx}\sin(x)\Big|_{x=\pi/3}$$

$$= \cos(x)\Big|_{x=\pi/3}$$

$$= \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}.$$

Example 4.6

Without the use of a CAS calculator, differentiate each of the following with respect to x:
(a) $\sin(5x)$ (b) $\cos(4x-1)$ (c) $\sin^5(2x)$ (d) $e^{\tan(x)}$ (e) $e^x\cos(3x)$

Solution:

(a) $y = \sin(5x) \Rightarrow \dfrac{dy}{dx} = 5\cos(5x)$

(b) $y = \cos(4x-1) \Rightarrow \dfrac{dy}{dx} = -4\sin(4x-1)$

(c) $y = \sin^5(2x) \Rightarrow \dfrac{dy}{dx} = 5\sin^4(2x)\times\cos(2x)\times 2$

$= 10\cos(2x)\sin^4(2x)$

(d) $y = e^{\tan(x)} \Rightarrow \dfrac{dy}{dx} = \sec^2(x)\times e^{\tan(x)}$

(f) $y = e^x\cos(3x) \Rightarrow \dfrac{dy}{dx} = e^x\cos(3x) + e^x[-\sin(3x)\times 3]$

$= e^x[\cos(3x) - 3\sin(3x)]$

Example 4.7

Given that $y = x\cos(x)$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

Solution:

$$y = x\cos(x)$$

Differentiating once: $\dfrac{dy}{dx} = \cos(x) - x\sin(x)$

Differentiating again: $\dfrac{d^2y}{dx^2} = -\sin(x) - [\sin(x) + x\cos(x)]$

$$= -2\sin(x) - x\cos(x)$$

Exercise 4.2

1. For each of the following limits, identify the function being differentiated. Hence, evaluate each of these limits.

(a) $\lim\limits_{h\to 0}\left[\dfrac{\sin(x+h)-\sin(x)}{h}\right]$ (b) $\lim\limits_{h\to 0}\left[\dfrac{\cos(y+h)-\cos(y)}{h}\right]$

(c) $\lim\limits_{h\to 0}\left[\dfrac{\tan(x+h)-\tan(x)}{h}\right]$ (d) $\lim\limits_{h\to 0}\left[\dfrac{(x+h)\sin(x+h)-x\sin(x)}{h}\right]$

(e) $\lim\limits_{h\to 0}\left[\dfrac{t\cos(t+h)-t\cos(t)}{h}\right]$ (f) $\lim\limits_{h\to 0}\left[\dfrac{\sin(\theta+\pi+h)-\sin(\theta+\pi)}{h}\right]$

2. Find the exact value of each of the following:

(a) $\lim\limits_{h\to 0}\left[\dfrac{\sin(\pi/4+h)-\sin(\pi/4)}{h}\right]$ (b) $\lim\limits_{h\to 0}\left[\dfrac{\sin(\pi/2+h)-1}{h}\right]$

(c) $\lim\limits_{h\to 0}\left[\dfrac{\cos(2+h)-\cos(2)}{h}\right]$ (d) $\lim\limits_{h\to 0}\left[\dfrac{\tan(\pi/3+h)-\sqrt{3}}{h}\right]$

(e) $\lim\limits_{h\to 0}\left[\dfrac{\sin(h)}{h}\right]$ (f) $\lim\limits_{h\to 0}\left[\dfrac{-\cos(h)+1}{h}\right]$

3. Differentiate each of the following with respect to x:

(a) $\sin(2x)$ (b) $\cos(x/4)$ (c) $\sin(\pi/2+x)$

(d) $\cos(3x)$ (e) $\cos(-x/2)$ (f) $\cos(\pi/4 - x)$

(g) $\tan(5x)$ (h) $\tan(x/5)$ (i) $\tan(\pi/4 + 2x)$

4. Differentiate each of the following with respect to θ:

(a) $\sin(\theta + 1)$ (b) $\cos(2\theta + 1)$ (c) $\tan(1 + \theta)$

(d) $\cos(1 - \theta)$ (e) $\tan(\theta/4 + 1)$ (f) $\sin[(\theta + 2)/4]$

(g) $2\cos(\theta + 4)$ (h) $[\sin(1 - \theta)]/2$ (i) $\tan(\pi/4)$

5. Differentiate each of the following with respect to x:

(a) $\sin^2(x+1)$ (b) $\sin^3(1-2x)$ (c) $\cos^3(3x)$

(d) $\cos^2(2x^2+1)$ (e) $\tan^2(1-3x)$ (f) $\tan^3(x^2+1)$

6. Differentiate each of the following with respect to t:

(a) $t\sin(t)$ (b) $t^2\cos(2t)$ (c) $(1+t)\tan(t^2)$

(d) $t\sin^2(t)$ (e) $t\sin^2(2t)$ (f) $t^2\cos^2(1+3t)$

7. Differentiate each of the following (either with respect to x or t):

(a) $\sin(\pi t)\cos(\pi t)$ (b) $\tan^2(\pi t)$ (c) $e^{2t}\cos(\pi t)$

(d) $e^{-\pi t}\cos(2t)$ (e) $e^{-\pi t}\sin(2t-\pi/6)$ (f) $e^{2\pi t}\cos^2(2t)$

8. Differentiate each of the following with respect to t:

(a) $\dfrac{\sin(2t-1)}{t}$ (b) $\dfrac{\cos(3-2t)}{t^2}$ (c) $\dfrac{e^{\cos(t)}}{t}$

(d) $\dfrac{e^{\sin(t)}}{\sin(t)}$ (e) $t^2 e^{\cos(\pi t)}$ (f) $\tan(e^{\tan t})$

9. Find the first and second derivatives of the following:

(a) $\sin^3(1+t)$ (b) $\cos^4(2t)$ (c) $\tan(1+t)$

(d) $e^{\sin 2t}$ (e) $\cos(1+e^t)$ (f) $\tan^2 t$

05 Applications of Differentiation I

5.1 Equation of tangent

- If $y = f(x)$ represents the equation of a curve, then,
  $f'(x)$ or $\dfrac{dy}{dx}$ or y' represents the *gradient function* of the curve.
- The numerical value of the gradient of the curve at any point is obtained by substituting the coordinates (usually, just the x-coordinate) into the equation for the gradient function.
- The gradient of the tangent to the curve at a point P on the curve is equal to the gradient of the curve at the point P. From this, the equation of the tangent may be easily obtained.

Example 5.1

Without the use of a CAS calculator, find the gradient of the curve $y = \dfrac{1}{\sqrt{1+2x}}$ at the point $(4, \dfrac{1}{3})$. Hence, find the equation of the tangent to the curve at this point.

Solution:

$$y = \frac{1}{\sqrt{1+2x}} = (1+2x)^{-\frac{1}{2}}$$

Gradient function is given by $\quad \dfrac{dy}{dx} = -\dfrac{1}{2}(1+2x)^{-\frac{3}{2}} \times 2$

$$= -\frac{1}{(1+2x)^{\frac{3}{2}}}$$

At $(4, \dfrac{1}{3})$, $x = 4$, $\quad \dfrac{dy}{dx} = -\dfrac{1}{27}$

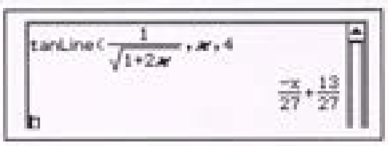
Hence, the gradient of the curve at (1, 5) is $-\dfrac{1}{27}$.

The gradient of the tangent to the curve at (1, 5) is $-\dfrac{1}{27}$.

The tangent passes through $(4, \dfrac{1}{3})$.

Hence, the equation of the tangent is $y - \dfrac{1}{3} = -\dfrac{1}{27}(x-4)$

$$\Rightarrow y = -\frac{x}{27} + \frac{13}{27}.$$



Exercise 5.1 *Unless otherwise stated avoid the use of a calculator.*

1. Find the gradient of the curve with equation $y = (1 - 2x)^5$ at the point (1, −1). Hence, find the equation of the tangent to the curve at (1, −1).

2. Find the equation of the tangent to the curve:
   (a) $y = \sqrt{9+x^2}$ at the point (4, 5) (b) $y = \dfrac{16}{(1+x)^2}$ at the point where $x = 3$.
   (c) $y = \dfrac{3}{\sqrt{5x-1}}$ at the point where $x = 2$ (d) $y = e^{2x}$ at the point where $x = 1$.

3. Find the equation of the tangent to the curve $y = e^{-2x}$ at the point where $x = -1$. Hence, find where this tangent intersects the x-axis.

4. The normal to a given curve at a point is a line that is perpendicular to the tangent at that point. Find the equation of the normal to the curve:
   (a) $y = x^2 e^x$ at the point $(1, e)$ (b) $y = e^x \sin x$ at the point where $x = 0$
   (c) $y = \dfrac{e^x}{2-x}$ at the point where $x = 1$ (d) $y = \dfrac{\cos x}{x}$ at the point where $x = \pi/2$.

5. The normal to the curve $y = e^{-x} \cos x$ at the point where $x = 0$, cuts the x-axis at the point P. Find the coordinates of the point P.

6. Find the coordinates of the point(s), on the curve $y = f(x)$, where the gradient is zero:
   (a) $y = \dfrac{1}{\sqrt{1+x^2}}$ (b) $y = \dfrac{x^2}{1-x}$ (c) $y = (1-x^2)e^{1+x^2}$ (d) $y = e^{\sin x}$ for $0 \le x \le \pi$

7. Find the point(s) on the curve $y = f(x)$ where the tangent to the curve at these points are parallel to the indicated lines:
   (a) $y = e^{2x}$, $y = 2x + 5$ (b) $y = \sqrt{1+x}$, $y = 1 + x/4$
   (c) $y = \dfrac{x^2}{x+2}$, $3x + y = 1$ (d) $y = x + e^{2x}$, $3x - y = 1$

8. $y = \dfrac{1}{2} - \dfrac{x}{2}$ is a tangent to the curve $y = \dfrac{a-x}{b+x}$, at the point (1, 0). Find a and b.

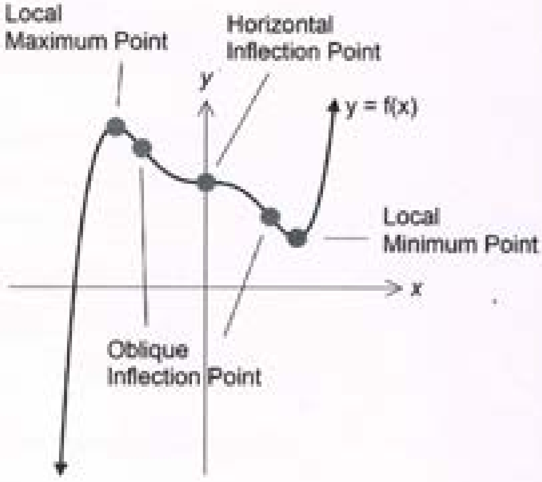
9. $y = x + 1$ is a tangent to the curve $y = ax + b \sin x$ at the point $(\frac{\pi}{2}, 1 + \frac{\pi}{2})$. Find a and b.

10. Use your CAS/graphic calculator, to find the equation of the tangent to the curve $y = (1+e^{x^2})\sin x$ at the point where $x = 0$. Hence, find the coordinates of the point(s) where this tangent meets the curve again for $-\pi < x \le \pi$.

5.2 Stationary Points and Inflection Points

Hands On Task 5.1

In this task, we will explore the mathematical properties of stationary and inflection points.



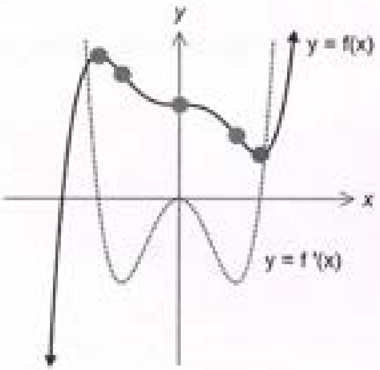
The accompanying diagram show the terms used to describe the stationary and inflection points of a curve. Stationary points consist of:

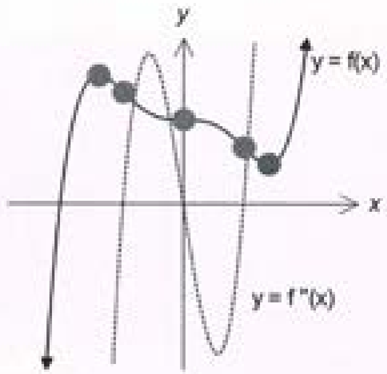
- minimum turning points
- maximum turning points
- horizontal inflection points.

Points of inflection consist of:

- horizontal inflection points
- oblique inflection points.

The two subsequent diagrams show the graphs of $y = f(x)$ with the graphs of $y = f'(x)$ and $y = f''(x)$.





1. Compare the graphs provided and complete the table below using the terms "zero" "negative" or "positive".

	Value of $f'(x)$	Value of $f''(x)$
Local minimum point		
Local maximum point		
Horizontal inflection point		
Oblique inflection point		

2. (a) Statement: If $y = f(x)$ has an inflection point at $x = a$, then $f''(a) = 0$.
   Use the association between inflection points on $y = f(x)$ and turning points on $y = f'(x)$ to determine why this statement has to be true.

   (b) Statement: When $f''(a) = 0$, then the curve $y = f(x)$ has an inflection point at $x = a$.
   Find a curve with equation $y = f(x)$ that disproves this statement.
   Find another condition, which together with $f''(a) = 0$, defines an inflection point.

△ Summary

- When $f'(a) = 0$, then the curve $y = f(x)$ has a stationary point at $x = a$.
  To identify the nature of the stationary point, one of two tests can be used.
  - The sign test (use this test when $f''(x)$ is difficult to obtain).
    - The stationary point at $x = a$, is a *maximum turning point* if :

x	$x = a^-$	$x = a$	$x = a^+$
sign for dy/dx or $f'(x)$	+ /	0 —	− \

    - The stationary point at $x = a$, is a *minimum turning point* if :

x	$x = a^-$	$x = a$	$x = a^+$
sign for dy/dx or $f'(x)$	− \	0 —	+ /

    - The stationary point at $x = a$, is a *horizontal inflection point* if :

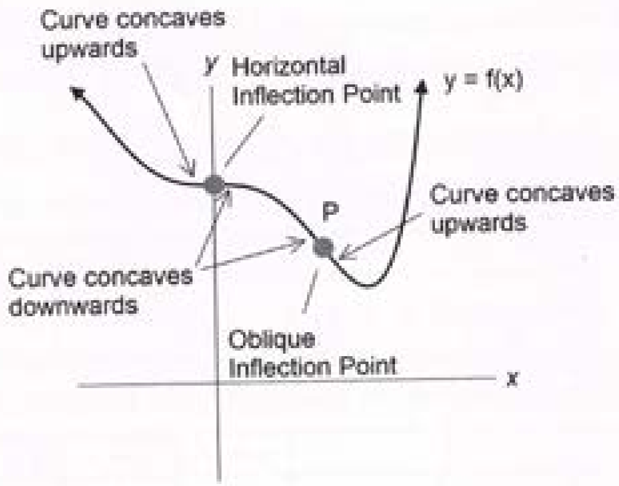
x	$x = a^-$	$x = a$	$x = a^+$
sign for dy/dx or $f'(x)$	− \ or + /	0 — 0 —	− \ + /

    a^- is a value of x slightly less than a and a^+ is a value of x slightly greater than a.

  - The second derivative test.
    - The stationary point is a *maximum turning point* if $f''(a) < 0$.
    - The stationary point is a *minimum turning point* if $f''(a) > 0$.
    - The stationary point is a *horizontal inflection point* if $f''(a) = 0$ and $f''(a^-)$ and $f''(a^+)$ have opposite signs.

- The curve $y = f(x)$ has an inflection point at $x = a$, when $f''(a) = 0$ and $f''(a^-)$ and $f''(a^+)$ have opposite signs.
  - If $f'(a) = 0$ at the same time, then the point is a point of horizontal inflection.
  - If $f'(a) \neq 0$ at the same time, then the point is a point of oblique inflection.

5.2.1 Inflection Points.



- An inflection point is technically defined as the point about which the curve undergoes a change of curvature.
- In the diagram above, note the change in concavity of the curve just before and after the inflection points.
- The curvature of the curve can be described by rate with which the gradient changes; $\frac{d}{dx}f'(x) = f''(x)$. Consider the inflection point P.
  - Just to the left of P, moving towards P,
    the gradient of the curve is becoming more negative.
    Hence, the rate of change of gradient $f''(x)$ is negative.
  - Just to the right of P, moving away from P,
    the gradient of the curve is becoming less negative.
    Hence, the rate of change of the gradient $f''(x)$ is positive.
  - That is, the rate of change of gradient changes from negative to positive (moving left to right of the point P).
    The inflection point P marks the point where the rate of change of gradient, $f''(x)$, is zero.
- If the point P has coordinates $(a, f(a))$, then:
  - $f''(a) = 0$
  - $f''(a^-) < 0$
  - $f''(a^+) > 0$.
- The change of sign for $f''(a^-)$ and $f''(a^+)$ is a necessary condition for an inflection point. This is because a curve like $y = x^4$ has $f''(0) = 0$ but (0, 0) is not an inflection point.

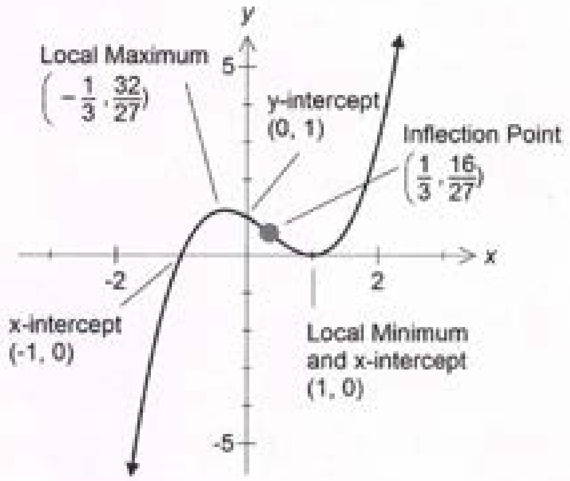
Example 5.2

Without the use of a CAS calculator, find the coordinates of the stationary and inflection points for $y = (x+1)(x-1)^2 \equiv x^3 - x^2 - x + 1$. Identify the nature of these points. Sketch this curve showing clearly all intercepts and turning points and inflection points.

Solution:

$$y = x^3 - x^2 - x + 1 \Rightarrow \frac{dy}{dx} = 3x^2 - 2x - 1$$

$$\text{and } \frac{d^2y}{dx^2} = 6x - 2.$$



For stationary points: $\frac{dy}{dx} = 0$.

$$\Rightarrow 3x^2 - 2x - 1 = 0$$
$$(3x+1)(x-1) = 0$$
$$x = -\frac{1}{3}, 1.$$

When $x = -\frac{1}{3}$, $y = \frac{32}{27}$, $\frac{d^2y}{dx^2} < 0$.

Hence, $(-\frac{1}{3}, \frac{32}{27})$ is a maximum point.

When $x = 1$, $y = 0$, $\frac{d^2y}{dx^2} > 0$.

Hence, $(1, 0)$ is a minimum point.

For inflection points, $\frac{d^2y}{dx^2} = 0$.

$$\Rightarrow 6x - 2 = 0$$
$$x = \frac{1}{3}$$

When $x = \frac{1}{3}$, $y = \frac{16}{27}$ and $\left.\frac{d^2y}{dx^2}\right|_{x=\left(\frac{1}{3}\right)^-} < 0$

and $\left.\frac{d^2y}{dx^2}\right|_{x=\left(\frac{1}{3}\right)^+} > 0$.

Hence, $(\frac{1}{3}, \frac{16}{27})$ is an oblique inflection point.

The sketch of $y = x^3 - x^2 - x + 1$ is drawn above.

Note:

- *In the previous unit, the natures of the stationary points were determined using the sign test. In this instance, the second derivative test is used instead. Notice that there is no significant "savings" in time.*
- *In the next example, the sign test becomes more efficient as the process of obtaining the second derivative is more arduous.*

Example 5.3

Use calculus to find the coordinates of the stationary point(s) and inflection point(s) of the curve $y = \dfrac{x^2}{1+x^2}$. Identify the nature of these points.

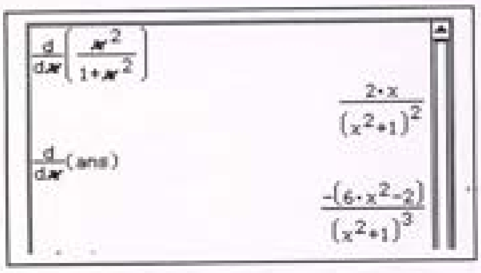
Solution:

$$y = \frac{x^2}{1+x^2} \Rightarrow \frac{dy}{dx} = \frac{(1+x^2)(2x) - x^2(2x)}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(1+x^2)^2(2) - 2x[4x(1+x^2)]}{(1+x^2)^4}$$

$$= \frac{(1+x^2)(2-6x^2)}{(1+x^2)^4}$$

$$= \frac{(2-6x^2)}{(1+x^2)^3}$$



For stationary points:

$$\frac{dy}{dx} = 0 \Rightarrow \frac{2x}{(1+x^2)^2} = 0 \Rightarrow x = 0.$$

When $x = 0$, $y = 0$, $\dfrac{d^2y}{dx^2} > 0$.

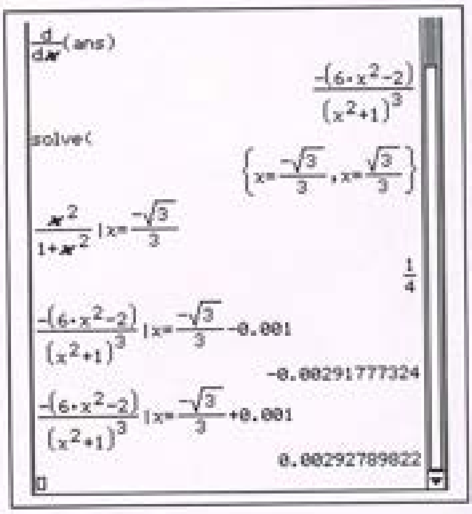
Hence, (0, 0) is a minimum turning point.

For inflection points:

$$\frac{d^2y}{dx^2} = 0 \Rightarrow \frac{(2-6x^2)}{(1+x^2)^3} = 0 \Rightarrow x = \pm\frac{\sqrt{3}}{3}.$$

When $x = -\dfrac{\sqrt{3}}{3}$, $y = \dfrac{1}{4}$, $\left.\dfrac{d^2y}{dx^2}\right|_{x=\left(-\frac{\sqrt{3}}{3}\right)^-} < 0$

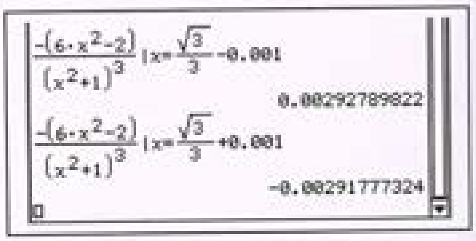
and $\left.\dfrac{d^2y}{dx^2}\right|_{x=\left(-\frac{\sqrt{3}}{3}\right)^+} > 0.$



Hence, $\left(-\dfrac{\sqrt{3}}{3}, \dfrac{1}{4}\right)$ is an oblique inflection point.

When $x = \dfrac{\sqrt{3}}{3}$, $y = \dfrac{1}{4}$, $\left.\dfrac{d^2y}{dx^2}\right|_{x=\left(\frac{\sqrt{3}}{3}\right)^-} > 0$

and $\left.\dfrac{d^2y}{dx^2}\right|_{x=\left(\frac{\sqrt{3}}{3}\right)^+} < 0.$



Hence, $\left(\dfrac{\sqrt{3}}{3}, \dfrac{1}{4}\right)$ is an oblique inflection point.

Exercise 5.2

1. Without the aid of a calculator, for each of the following curves:
   (i) where possible, use the second derivative test to identify the nature of the stationary points
   (ii) find the coordinates of the inflection point(s), where they exist,.

   (a) $y = x^3 - 3x^2 + 3x + 1$ (b) $y = 6 - 12x + 6x^2 - x^3$
   (c) $y = 2x^3 - 4x^2 + 2x$ (d) $y = x^4 + x^3$

2. Use calculus to find the coordinates of the stationary and inflection points, where they exist, for each of the following curves.

   (a) $y = xe^x$ (b) $y = \frac{e^x}{x}$ (c) $y = \frac{x^2}{x^2 - 1}$ (d) $y = \frac{2x}{1 - x^2}$

3. Use calculus to find the coordinates of the stationary point(s) for $y = x \sin x$ for $0 \le x \le 2\pi$. Use a calculus method to identify the nature of these points.

4. Use calculus to find the coordinates of the stationary point(s) and inflection point(s) for $y = e^x \cos x$ for $-\pi < x \le \pi$. Use a calculus method to identify the nature of these points.

5. The curve $y = ax^3 + 2x^2 + bx + 2$ has a turning point at $x = 1$ and an inflection point at $x = -1$. Find the values of a and b.

6. The curve $y = ax^3 + bx^2 + 2x - 1$ has a turning point at $x = -1$ and an inflection point at $x = 2$. Find the values of a and b.

7. The curve $y = ax^4 + bx^3 - x^2 + 1$ has an inflection point at $(1, -4)$. Find a and b.

8. The curve $y = \frac{ax}{(x+b)^2}$ (where $a \neq 0$) has a stationary point at the point where $x = 1$ and an inflection point at the point where $x = 2$. Find a and b.

9. Without the use of a calculator, sketch each of the following curves. Indicate clearly all intercepts, turning points and asymptotes where they exist.

   (a) $y = x(x-3)^2$ (b) $y = x^4 + 2x^3$ (c) $y = \frac{x^5}{5} - \frac{4x^3}{3}$
   (d) $y = xe^{2x}$ (e) $y = \frac{e^x}{x^2}$ (f) $y = \frac{x^2}{e^x}$

10. Sketch the polynomial curve with equation $y = f(x)$:
    (a) which has exactly one turning point at $(1, -2)$ and inflection points at $(0, 0)$, $(0.6, -1)$, $(1.\ 5, -1)$ and $(2, 0)$.
    (b) which has the only turning points at $(1.4, 8.3)$ and $(-0.6, -1.1)$ and inflection points at $(-2, 0)$, $(-1.1, -0.6)$ and $(0.7, 4.6)$.

06 Applications of Differentiation II

6.1 Instantaneous Rate of Change and Optimisation

- For $y = f(x)$, the derivative $\frac{dy}{dx}$ measures the *instantaneous rate with which y changes with respect to x.*
- Hence, $\frac{dy}{dx}$ may be viewed as an "instrument" to measure and describe the rate of change between the variables x (independent variable) and y (dependent variable).
- The dependent variable achieves a maximum or minimum value when the "rate measurer" is numerically zero. We can use this idea to find the maximum and minimum values of functions.

Example 6.1

Given that $A = \frac{0.4t}{t^2+8}$, find using analytical methods:

(a) the exact value of the instantaneous rate of change of A with respect to t when $t = 1$

(b) the maximum value (exact) of A and the corresponding value of t

(c) the average rate of change of A for $0 \le t \le 1$.

Solution:

(a) Rate of change $\quad \frac{dA}{dt} = \frac{(t^2+8)(0.4) - 0.4t(2t)}{(t^2+8)^2} = \frac{-0.4t^2+3.2}{(t^2+8)^2}$

When $t = 1$, $\quad \frac{dA}{dt} = \frac{-0.4(1)^2+3.2}{(1^2+8)^2} = \frac{14}{405}$

(b) For max. value $\quad \frac{dA}{dt} = 0 \Rightarrow \frac{-0.4t^2+3.2}{(t^2+8)^2} = 0$

$-0.4t^2 + 3.2 = 0 \Rightarrow t = \pm\sqrt{8}$

Using the sign test:

t	$(\sqrt{8})^-$	$\sqrt{8}$	$(\sqrt{8})^+$
sign of dy/dx	+	0	–

Hence, A is maximised when $t = \sqrt{8}$. Maximum value for $A = \frac{\sqrt{2}}{20}$.

(c) Average rate of change $= \frac{A(1) - A(0)}{1} = \frac{\frac{0.4}{9} - 0}{1} = \frac{2}{45}$

Exercise 6.1

1. Organic waste, when deposited in a lake, decreases oxygen content of the water. If t denotes time in days after the waste is deposited, then it is found experimentally that the oxygen content is given in a particular instance by $y = t^3 - 30t^2 + 6\,000$ for $0 \le t \le 25$.
   (a) Find an expression for the instantaneous rate of change of oxygen content.
   (b) Show how your answer in (a) can be used to find the maximum and minimum values of y during the first 25 days following the deposition of the waste.

2. The size of a population of bacteria that is introduced to a nutrient, grows according to the formula $P(t) = 5\,000 + \dfrac{3\,000t}{100 + t^2}$, where t is time measured in hours.
   (a) Find an expression for the instantaneous rate of change of the bacteria population.
   (b) Hence, determine the maximum size of the population.

3. During an influenza epidemic, the proportion of the population in a particular suburb who are infected is denoted $p(t)$ where t is the time in weeks after the start of the epidemic. Given that $p(t) = t/(4 + t^2)$: (a) describe in words dp/dt
   (b) find when most of the population in the suburb is infected and the maximum proportion of population affected.

4. The concentration, C mg/kL, of a chemical in a lake, at time t weeks is given by $C = 0.2(1 + 8t)\,e^{-t}$, for $0 \le t \le 8$. Find:
   (a) the exact value of t when the instantaneous rate of change of C with respect to t is 0
   (b) the exact maximum concentration of the chemical and state when this occurs.

5. The displacement of a particle, y cm, from its equilibrium position, is given by $y = 10\,e^{-t}\sin(t)$, for $0 \le t \le \pi$, where t is time in seconds. Find:
   (a) an expression for the instantaneous rate of change of y with respect to t
   (b) the exact maximum displacement of the particle and state when this occurs.

6. The east-west cross-sectional profile of a gully is modelled by $h = -x\,e^{-0.05x^2}$, for $x > 0$, where h is the depth of the gully (in metres) and x is the horizontal distance from the western end of the gully.
   (a) Find an expression for the rate of change of h with respect to x.
   (b) Find the exact depth of the deepest part of the gully and the corresponding x value.

7. For $2 \le t \le 6$, the population of a certain bacteria in a culture (in hundreds) is modelled by $P = t + \sin(2t)$, where t is time in weeks.
   (a) Find the exact value of t when the bacteria achieve a temporary peak in its population. Give the population at this time.
   (b) Find the maximum population of the bacteria in the interval $2 \le t \le 6$.

8. The number of reported cases , P hundred, of a particular throat infection is modelled by $P = 10 - t - \cos(1.2t)$ for $0 \le t \le 9$, where t is time in weeks. Use a calculus technique to find the times when the number of reported cases of the infection was temporarily at a minimum. Give the corresponding number of cases for each of these times.

9. The temperature θ^{0}C inside a garden shed t hours after 6 am is modelled by the equation $\theta = 25 + 10 \sin\left(\frac{\pi t}{12} - \frac{\pi}{4}\right)$. Use derivatives to determine the:
   (a) rate of temperature increase at noon
   (b) maximum temperature inside the shed and the first time after 6 am when this occurs
   (c) when the temperature is increasing at a rate of $\frac{5\pi}{12}$ degrees Celsius per hour.

10. The water depth, h meters, measured from the bottom of a lake, t hours after 6 am is modelled by the equation $h = 15 - 2\cos\left(\frac{\pi t}{6} - \frac{\pi}{12}\right)$. Use calculus to determine:
    (a) the rate with which the water depth is changing at 6 pm
    (b) the minimum water depth and the first time after 6 am when this occurs
    (c) the time when the depth is decreasing at a rate of 1 m per hour.

11. The displacement, x cm, of a particle from a fixed point O, t seconds after it is released is modelled by the equation $x = -5\cos\frac{\pi t}{4}$. Use a calculus method to determine:
    (a) the velocity of the particle after 2 seconds
    (b) when during the interval $0 \le t \le 8$, the particle travels with a speed of 1 cms^{-1}.

12. The value of a certain fruit crop ($) is given by $V = a(1 - e^{-bS})$ where a and b are positive constants and S is the number of kilograms per hectare with which the crop is sprayed. If the cost of spraying is given by $C = \lambda S$ where λ is a constant, find the value of S that makes $V - C$ a maximum. Interpret $ab < \lambda$.

13. The speed with which a kangaroo can jump depends on the angle x at which it leaves the ground. Assume that the horizontal velocity of a hopping kangaroo is modelled by the equation $V = a\cos(x)\,[1 + \sin(x)]$, where a is a constant. If the kangaroo travels by continuous hopping, find the angle to the horizontal with which it must hop each time to achieve maximum horizontal speed. Find this maximum speed.

6.1.1 More Optimisation

Example 6.2

Given that $A = x + 2y$ and $xy = 5$ where $x > 0$, use a calculus technique to find the exact minimum value of A and the exact values of x and y when this occurs.

Solution:

$$A = x + 2y$$
$$= x + \frac{10}{x} \quad \text{since } xy = 5 \Rightarrow y = \frac{5}{x}$$
$$\frac{dA}{dx} = 1 - \frac{10}{x^2} \quad \text{and} \quad \frac{d^2A}{dx^2} = \frac{20}{x^3}.$$

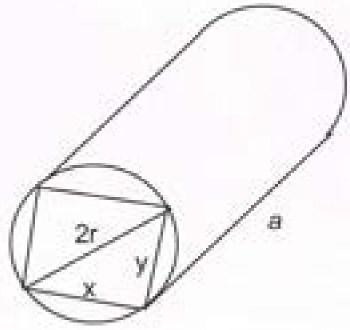
For max/min values $\frac{dA}{dx} = 0 \Rightarrow 1 - \frac{10}{x^2} = 0 \Rightarrow x = \sqrt{10}$ (since $x > 0$)

When $x = \sqrt{10}$, $\frac{d^2A}{dx^2} > 0$. Hence, A is minimised when $x = \sqrt{10}$.

Minimum value of $A = 2\sqrt{10}$ when $x = \sqrt{10}$ and $y = (\sqrt{10})/2$.

Example 6.3

Given a log of length a with uniform circular cross-section of radius r, use calculus techniques to find the dimensions of a piece of plank of rectangular cross-section that can be cut from this log such that the volume of the plank is a maximum.



Solution:

Let x and y be the width and height of the rectangular cross-section of the plank respectively.

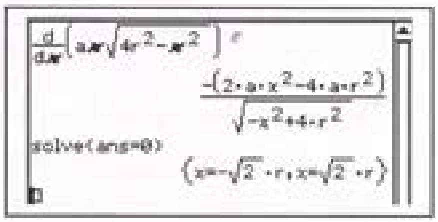
Hence, volume of plank, $V = axy$

Using Pythagoras' Theorem on the rectangular cross-section:

$$x^2 + y^2 = (2r)^2 \Rightarrow y = (4r^2 - x^2)^{1/2}$$

Therefore $V = ax\sqrt{4r^2 - x^2}$.

$$\frac{dV}{dx} = \frac{2a(2r^2 - x^2)}{\sqrt{4r^2 - x^2}}$$



For max/min values $\frac{dV}{dx} = 0$. $\quad \frac{2a(2r^2 - x^2)}{(4r^2 - x^2)^{\frac{1}{2}}} = 0$

$$\Rightarrow x = r\sqrt{2} \quad (x > 0)$$

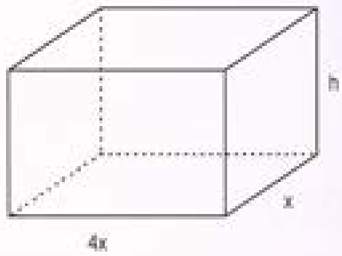
Using the sign test:

x	$(r\sqrt{2})^-$	$(r\sqrt{2})$	$(r\sqrt{2})^+$
dV/dx	+	0	–

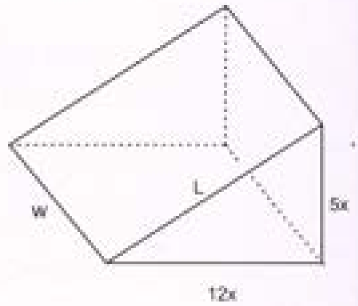
This confirms that V is maximised at $x = r\sqrt{2}$ with $y = r\sqrt{2}$.

Exercise 6.2

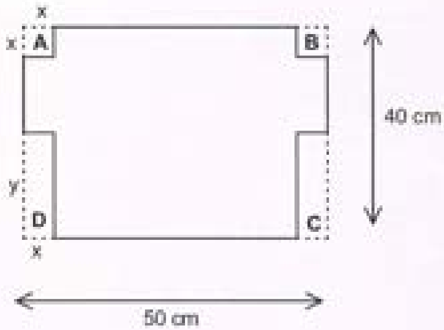
1. A piece of wire, 400 cm long, is used to make the 12 edges of the frame of a rectangular box. The length of the rectangular frame is 4 times that of the width of the frame, x cm. The height of the box is h.
   (a) Find h in terms of x.
   (b) Find the volume, V, of the box in terms of x.
   (c) Use a calculus technique to find the exact dimensions of the frame that will maximise the volume of the box. Give this volume.



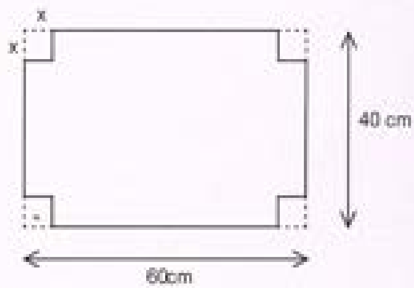
2. A piece of wire, 500 cm long, is used to make the 9 edges of the frame of a regular wedge. The height and length of the wedge are $5x$ and $12x$ respectively. L is the length of the hypotenuse of the cross-section.
   (a) Find L in terms of x.
   (b) Find the width of the wedge, w, in terms of x.
   (c) Find V, the volume of the wedge, in terms of x.
   (d) Use a calculus technique to find the exact dimensions of the frame that will maximise the volume of the wedge. Give this volume.



3. A rectangular sheet of cardboard, 40cm by 50 cm, is to be made into a closed rectangular box. A square, each of side, x cm, is removed from each of the corners A and B of the cardboard. A rectangle, each of dimensions x cm by y cm, is removed from each of the corners C and D of the cardboard.
   (a) Find the length, L, of the box in terms of x.
   (b) Show that the width of the box is given by $w = 20 - x$.
   (c) Find the volume, V, of the box, in terms of x.
   (d) Use a calculus technique to find the dimensions of the box that will maximise its volume. Find this volume.



4. A rectangular sheet of cardboard, 40 cm by 60 cm, is to be made into an open rectangular box. Four squares, each of side, x cm, are removed from each corner of the cardboard. Use a calculus technique to find the exact maximum volume of the box. Give the corresponding dimensions of the box.



5. The total surface area of a closed cylindrical container is 2 400 cm^2. Find the exact dimensions of the container that will maximise its volume and give this maximum volume.

6. A closed rectangular box, has a volume of 15 000 cm^3. The height of the box is twice its width. Find the exact dimensions of the box that will minimise its surface area.

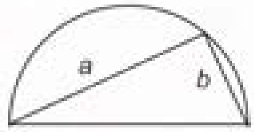
7. A cylindrical canister, closed at both ends, has a volume of 24 000 cm^3. Find the minimum surface area of the canister. Give the corresponding dimensions of the canister.

8. The interior of a playing field consists of a rectangle of length L and width $2r$, with semicircles of radius r at two opposite ends. The perimeter of the field is 1 500 m. Use a calculus technique to find the dimensions that will maximise the area of the playing field. Give this area.

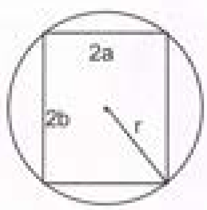
9. A portion of a backyard in the shape of a sector of a circle is to be fenced with 100m of fencing. Find the angle of the sector which maximises the area to be fenced. Find this maximum area.

10. The area of a sector of a circle of radius r is 50 cm^2. Use a calculus technique to find the least possible perimeter of the sector.

11. A right angled triangle with sides of length a, b and $2r$ is trapped within a semicircle of fixed radius r. Given that all three vertices are on the semicircle, find the values of a and b that will maximise the area of the triangle.



12. A rectangle of length $2a$ and width $2b$ is trapped within a circle of fixed radius r. Given that all four vertices of the rectangle are on the circle, find the value of a and b that will maximise the area of the rectangle.



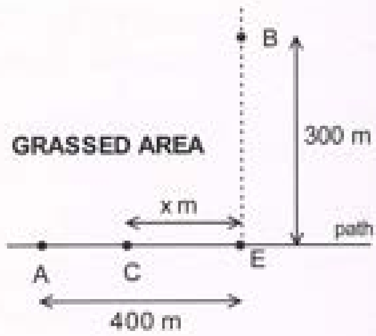
13. The radius of a circular wire frame is 0.5 m. The frame is cut and is used to make two new circular frames. There is no "wire" left behind. Find the radius of each of these new frames so that the total area of these two circles is a minimum.

14. The length and width of an open rectangular tank of volume 32m^3 is respectively x and y. Ignore the thickness of the material.
   (a) Show that the surface area of the tank is given by $A = xy + 64(1/x + 1/y)$.
   (b) Prove that if y is fixed and x is allowed to vary, the minimum value of A is $16\sqrt{y} + 64/y$.
   (c) If, however, x is fixed and y is allowed to vary, find the minimum value of A.

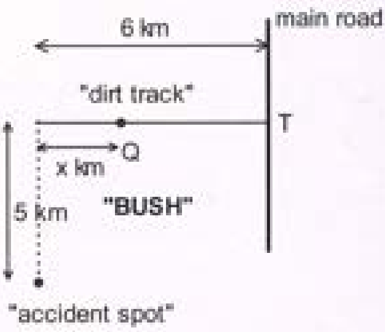
15. The amount of heat lost from a closed flask that is filled with hot water is found to be directly proportional to the surface area of the flask. Given that the flask is cylindrical in shape with flat ends and constant volume, find the ratio between the length of the flask and the radius of its cross-section that will minimise the amount of heat lost.

16. A truck is required to convey its cargo from town A to town B, a distance of 1 200 km. Fuel costs is equal to the cube of the square root of the speed of the truck. Variable costs are pegged at \$50 per hour, with fixed costs of \$200. Find the speed, v km/h, that will minimise the cost of the operation.

17. Jane needs to get from A (a point on a footpath) to B (a point in the grassed area). Jane can walk at a steady speed of 100 m/min along the footpath AE. She can walk at a steady speed of 60 m/min across the grassed area. Given that AE is perpendicular to EB, with AE = 400 m and EB = 300 m. Jane decides to walk along the footpath from to C, x m from E, and then from C directly to B across the grassed area. Find the value of x so that the time Jane takes to walk from A to B via C (across the grassed area) is a minimum. Give this minimum time.



18. Cindy can ride her mountain bike on a dirt track at a speed of 12 km per hour. In the "bush" her speed on the mountain bike is 4.8 km per hour. Cindy and her friend Bob were out bike-trekking in the bush when Bob fell from his bike and badly sprained his ankle. Cindy will have to leave Bob at the scene of the accident and ride her bike to the junction T to seek help. Cindy can either ride directly to junction T or ride to a point Q on a dirt track and then along the dirt track to junction T. Describe the route Cindy should have taken to minimise the total journey time and give this minimum time.



19. The accompanying diagram shows the cross-section of a trough of length 5 m. The cross-section is a trapezium such that AB is parallel to DC. AD = BC = 40 cm and AB = 50 cm. $\angle ADC = \angle BCD = \theta$ radians. Find the value of θ that will maximise the volume of the trough.

20. Two points P and Q are located on the same horizontal plane and are at a distance a apart. An object J initially at Q falls vertically downwards such that at time t its vertical distance from Q is given as kt^2, where k is a constant. At time t, the angle of depression of J from P is θ. (a) Prove that $\dfrac{d\theta}{dt} = \dfrac{2akt}{a^2 + k^2t^4}$.

(b) Find at which instance, $d\theta/dt$ is maximised, and give the value of θ then.

*21. A cone with semi-vertical angle x is trapped in a sphere of radius r.

(a) Show that the volume of the cone is $\dfrac{8}{3}\pi r^3 \sin^2(x)\cos^4(x)$.

(b) Find the curved surface area of this cone.

(c) If r is fixed and x allowed to vary:

(i) show that the maximum volume of this cone is 8/27 of the volume of the sphere

(ii) prove that the volume and the curved surface area of the cone are maximised at the same time.

6.2 Small Changes/Increments

- Using the notation δx to represent a small change in x, and δy to represent a small change in y, we can write $\dfrac{dy}{dx} = f'(x) = \lim\limits_{\delta x \to 0}\left[\dfrac{f(x+\delta x)-f(x)}{\delta x}\right] = \lim\limits_{\delta x \to 0}\left[\dfrac{\delta y}{\delta x}\right]$.
- Hence, as x changes from x_0 to $x_0 + \delta x$ and δy is the corresponding change in y, *if δx is sufficiently small*, then $\dfrac{\delta y}{\delta x} \approx \left.\dfrac{dy}{dx}\right|_{x = x_0,\, y = f(x_0)}$

  That is, the ratio of the corresponding changes in y and x can be *approximated* by the value of the gradient function evaluated at the original values of x and y, $(x_0, f(x_0))$.

Example 6.4

If $y = \tan(x)$, use derivatives to find the approximate change in y when x changes from 1.00 radians to 0.95 radians.

Solution:

$$y = \tan(x)$$

Hence, $$\frac{dy}{dx} = \sec^2(x)$$

Let δx and δy represent the incremental change in x and y respectively.

Initial value of $x = 1$. $\delta x = 0.95 - 1.00 = -0.05$

For small δx, $$\frac{\delta y}{\delta x} \approx \left.\frac{dy}{dx}\right|_{x=1}$$

$$\approx \left.\frac{1}{\cos^2(x)}\right|_{x=1}$$

$$\approx 3.42552$$

But $\delta x = -0.05$: $\Rightarrow$ $\delta y = 3.42552 \times -0.05 = -0.1713$

Exercise 6.3

1. Given that $y = x^3 + 1/x$, use derivatives to find the approximate change in y when x changes from 1.00 to (a) 1.01 (b) 0.95.
2. Given that $y = \sin(2x)$, use derivatives to find the approximate change in y when x changes from 2.000 to (a) 2.001 (b) 1.995.
3. Given that $y = x^2 e^x$, use derivatives to find the approximate change in y when x changes from (a) 1.5 to 1.55 (b) 1.5 to 1.49.

4. Given that $y = x\cos(x)$, use derivatives to find the approximate change in y that corresponds to x changing from (a) 2 to 2.01 (b) 2 to 1.97.

5. Use derivatives to find the approximate change in the surface area of a spherical balloon corresponding to a change in its radius from 4.00 cm to 4.01 cm.

6. Use derivatives to find the approximate change in the radius of a spherical balloon corresponding to a change in its surface area from 1 500 cm^2 to 1 490 cm^2.

7. Use derivatives to find the approximate change in the volume of a cylinder of height 10.0 cm corresponding to a change in its radius from 5.00 cm to 4.99 cm.

8. Use derivatives to find the approximate change in the base radius of a closed cylinder of height 5.0 cm corresponding to a change in its surface area from 200 cm^2 to 202 cm^2.

9. Use derivatives to find the approximate change in the curved surface area of a cone of height 10 cm corresponding to a change in its base radius from 3.00 cm to 3.02 cm.

10. A closed cylinder 10cm long has a cross-sectional radius of 2cm. Use derivatives to find the approximate error in the volume of the cylinder corresponding to an error of (a) 0.1cm in the measurement of its length, (b) 0.05cm in the measurement of its radius.

11. For the cylinder in Question 10, use derivatives to calculate the approximate error in its total surface area.

12. The height of a lamp post is estimated by measuring the length of the shadow cast by a 2 m long vertical pole placed 10 m away from the base of the lamp post. Assume that the "lamp" is at the top of the post. The shadow was measured to be 2.00 m with an error of 1 cm. Use derivatives to calculate the error in the height of the lamp post.

13. Given that $y = xe^x$, use derivatives to approximate the percentage error in y corresponding to a change in x from (a) 2.00 to 2.02 (b) 2.00 to 1.96.

*14. Use derivatives to calculate the approximate percentage error in the volume and surface area of a sphere of radius 4cm corresponding to an error of 1% in its radius.

*15. The period of oscillation of a pendulum of length L, is given by $T = 2\pi\sqrt{\frac{L}{g}}$, where g is a constant. Find the approximate percentage change in T corresponding to a 5% drop in the length of the pendulum. Find the approximate percentage change in L corresponding to a 2% increase in the period of oscillation.

6.3 Marginal Value

- Let $C(x)$ be the cost associated with the production of x items.
  To examine the effect on $C(x)$ caused by a change in x from x_0 to $x_0 + \delta x$, we consider,

$$\frac{\delta C}{\delta x} \approx \left.\frac{dC}{dx}\right|_{x = x_0, C = f(x_0)}$$

  If $\delta x = 1$, then $\delta C \approx \left.\frac{dC}{dx}\right|_{x = x_0, C = f(x_0)}$.

- δC is the change in cost associated with the production of one additional item based on an original production figure of x_0.
  That is, it is the additional cost of producing one additional item when x_0 have already been produced.
  Economists call this the *marginal cost* of the production.

- Mathematically, the marginal cost is then the value of the derivative at the original production figure.

- We can define *marginal revenue* and *marginal profit* in similar terms.

Example 6.5

The cost (in \$) of producing x items of a certain product is given by $C(x) = 1\,000 + 50x - 0.05x^2$. The product is sold at a price of \$49 per item. Find:
(a) the marginal profit of producing and selling 200 items
(b) the average profit for producing 200 items.

Solution:

(a) The profit function $P(x) = 49x - (1\,000 + 50x - 0.05x^2)$
$= -1\,000 - x + 0.05x^2$.

The marginal profit in producing and selling 200 items is:

$$\delta P \approx \left.\frac{dP}{dx}\right|_{x = 200}$$

$$= -1 + 0.1x\Big|_{x = 200}$$

$$= \$19$$

(b) $P(200) = \$800$ Hence, the average profit is \$800/200 = \$4.

Exercise 6.4

1. The cost, \$C, of producing x items of a product, is given by, $C = 2\,000 + 5x + 0.003x^2$. Use derivatives to find the marginal cost when 200 items have already been produced. Find the cost associated with the production of the 201st item.

2. The cost of producing x items of a product, is given by \$$(7\,000 + 20x)$. Each item is sold for \$$(60 - 0.01x)$. Use derivatives to find the marginal profit when 400 items have already been produced and sold and the profit associated with the sale of the 401st item.

3. The cost of producing x items of a product, is given by \$$[5x + 2000e^{-0.01x}]$. Each item is sold for \$24.90. Use derivatives to find the profit associated with the sale of the 501st item.

4. The cost (in \$) of producing x units of a product is given by $C(x) = 8\,000 + 3x + 40\sqrt{x}$. Find the marginal and average cost for producing 1 000 items.

5. The cost (in \$) of producing x units of a product is given by $C(x) = 25 + 5x^2 - 0.2x^3$. Find an expression for the marginal cost and find the value of x that will maximise the marginal cost.

6. The cost and revenue function (in \$) of manufacturing and selling x units of a product is given by $C(x) = 3.20 + 1.40x$ and $R(x) = 0.1x^2 - 0.001x^3$ respectively for $0 \le x \le 80$. Find the marginal cost and marginal profit for producing and selling x units. Find the value of x that will maximise the profit and marginal profit.

7. The cost function (in \$) of manufacturing x units of a product is given by: $C(x) = 5\,000 + 100x - 2x^2 + 0.01x^3$ for $0 \le x \le 150$. The product is sold at \$100 each. Find the value of x that will maximise the profit and marginal profit.

8. The selling price of an item when x items are produced ($0 \le x < 899$) is given by $p(x) = \$(900/(x + 1) - 1)$. Find the value of x that maximises the revenue and the marginal revenue when this occurs.

9. A small cottage factory sells a line of goods at a price of \$$(-0.001x^2 + 15)$ each, where x is the number of items produced each week. The total weekly cost is $-0.01x^2 + 3x + 200$. Find the maximum marginal profit and the value of x when this is achieved.

10. The selling price of an item is given by $p(x) = -0.002x^2 + 5x$, where x is the number of items manufactured. The cost of producing x items is given by $C(x) = 11x/10 + 3$. Determine how many items need to be produced to obtain the maximum marginal profit.

6.4 Growth and Decay II

- This section builds on the material covered in Chapter 1.

Hands On Task 6.1

In this task, we will rewrite an exponential growth/decay equation using derivatives.

1. The population P, of a colony of bacteria, at time t hours, is modelled by $P = 10\,e^{0.05t}$.
   (a) Show that the instantaneous rate of population growth is given by $\dfrac{dP}{dt} = 0.5\,e^{0.05t}$.
   (b) Show that the instantaneous rate of population growth can be written as $\dfrac{dP}{dt} = 0.05P$.
   (c) Interpret the statement $\dfrac{dP}{dt} = 0.05P$.

2. The population P, of a colony of bacteria, at time t hours, is modelled by $P = 100e^{-0.05t}$.
   (a) Show that the instantaneous rate of population growth is given by $\dfrac{dP}{dt} = -0.05P$
   (b) Interpret the statement $\dfrac{dP}{dt} = -0.05P$.

3. Let P be the population of a colony of bacteria at time t hours. The initial population is 100 and $\dfrac{dP}{dt} = 0.02P$.
   (a) Review Questions 1 and 2, to suggest an exponential expression for P in terms of t.
   (b) Verify that your expression for P is correct by finding $\dfrac{dP}{dt}$ and then rewriting $\dfrac{dP}{dt}$ in terms of P.

4. Repeat Question 3 for an initial population of 50 with $\dfrac{dP}{dt} = -0.04P$.

5. Let P be the population of a colony of bacteria at time t hours. The initial population is P_0 and $\dfrac{dP}{dt} = kP$. Suggest an exponential expression for P in terms of t.

6. Let P be the population of a colony of bacteria at time t hours. The initial population is P_0 and $\dfrac{dP}{dt} = -kP$. Suggest an exponential expression for P in terms of t.

7. Given that $\dfrac{dP}{dt} = kP$, suggest what k represents.

△ Summary

Let P be the population of a colony at time t, with an initial population of P_0.

- If the instantaneous rate of population growth is directly proportional to its population, then the population is said to experience exponential growth/decay.
  That is, if $\dfrac{dP}{dt} \propto P$ or $\dfrac{dP}{dt} = kP$,
  then the population experiences exponential growth/decay.
  k, the constant of proportionality is called the continuous percentage growth rate.
- If $\dfrac{dP}{dt} = kP$, then $P = P_0 e^{kt}$.
  If $k > 0$, then the population is growing. If $k < 0$, then the population is declining.

Example 6.6

A colony of feral cats grows in such a way that its population growth is proportional to its population. There were 50 cats at the start of 2005 and the cat population has a continuous percentage growth rate of 2% per year. .

(a) Find the population at the start of 2009.
(b) Find the instantaneous rate of population growth at the start of 2009.
(c) Find the rate with which the instantaneous growth rate is changing.

Solution:

Let P be the population t years after the start of 2005.
Then, $\quad P = 50\,e^{0.02t}$

(a) At the start of 2009, $t = 4$.
Hence, at the start of 2009, $\quad P = 50\,e^{0.02(4)} = 54.16 = 54$
Hence, there were 54 cats at the start of 2009.

(b) The instantaneous population growth rate is given by:

$$\frac{dP}{dt} = 0.02(50\,e^{0.02t}) = e^{0.02t}$$

Hence, when $t = 4$, $\quad \dfrac{dP}{dt} = e^{0.02(4)} = 1.08$

That is, at the start of 2009, the instantaneous growth rate is 1.08 cats per year.

(c) The rate with which the instantaneous growth rate changes is given by:

$$\frac{d^2P}{dt^2} = 0.02\,e^{0.02t}$$

Exercise 6.5

1. Australia's population (in millions), t years after 1995, is modelled by $P = 17.9\,e^{0.0078t}$.
   (a) State the continuous percentage population growth rate.
   (b) Find an expression for the instantaneous population growth rate.
   (c) Find the instantaneous population growth rate 10 years after 1995.
   (d) Find when the instantaneous population growth rate has a value of 0.2

2. The instantaneous population growth rate of a colony of feral goats is proportional to its population. There were 100 goats at the start of the study and the goat population grows continuously at a rate of 5% per year.
   (a) Find the population 3 years after the start of the study.
   (b) Find the instantaneous population growth rate after 10 years.
   (c) Find when the instantaneous population growth rate has a value of 10 goats per year.

3. The population of a colony of koalas grows exponentially with a continuous percentage growth rate of −0.5% per year, with an initial population of 800.
   (a) Find the population after 5 years.
   (b) Find the instantaneous population growth rate after 5 years.
   (c) Find when the instantaneous population growth rate has a value of −2 koalas per year.

4. A radioactive substance decays at a continuous rate of 2%. Initially, there were 100g of this substance.
   (a) Find the amount of this substance left after 100 years
   (b) Find the instantaneous decay rate after 100 years.
   (c) Find the rate with which the instantaneous decay rate changes after 100 years.

5. The population of a mountain tribe at the start of 2006 was 250. The population growth of this tribe is modelled by $dP/dt = 0.03P$, where P is the population t years after 2006.
   (a) Find P in terms of t.
   (b) Find when the instantaneous population growth rate is 10 persons per year.
   *(c) Find when the instantaneous population growth rate changes at a rate of 0.3 persons per year per year.

6. The population of a river tribe at the start of 2010 was 400. The population growth of this tribe is modelled by $dP/dt = -0.08P$, where P is the population t years after 1970.
   (a) Find the population after 10 years.
   (b) Find the initial instantaneous population growth rate.
   (c) Find the time taken for the instantaneous decay rate to be halved.

7. The instantaneous growth rate of a bacterial culture is 5% of its population size. Its initial population is 200.
   (a) Find the population size after 10 hours.
   (b) Find instantaneous population growth rate after 10 hours.
   (c) Find when the instantaneous growth rate doubles.

8. The instantaneous decay rate of a radioactive substance is 25% of the amount present. Initially there were 100g of this substance.
   (a) Find the amount of this substance left after 25 years.
   (b) Find instantaneous decay rate after 25 years.
   (c) Find the rate with which the decay rate changes after 25 years.

9. The amount of radioactive substance, X, remaining t years after 2010, is modelled by, $A = 50\,e^{-0.05t}$. The amount of radioactive substance, Y, remaining t years after 2010, is modelled by, $A = 60\,e^{-0.07t}$.
   (a) Find the instantaneous rate of growth of X and Y.
   (b) Find when the instantaneous rates of decay are the same.

10. The population of China, P million, t years after 1981, is modelled by, $P = 991\,e^{0.01t}$. The population of India, P million, t years after 1981, is modelled by, $P = 690\,e^{0.023t}$.
   (a) Find when the population of India first exceeds that of China.
   (b) Find when the instantaneous population growth rate of India is double that of China.

07 Anti-Differentiation

7.1 Anti-Differentiation of Polynomials

- Listed below, are the rules for anti-differentiating x^n for integer $n \geq 1$ and $m \geq 1$ as encountered in previous units.

> - $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + C$ where C is a constant.
> - $\int ax^n \, dx = a\int x^n \, dx$ where a and C are constants.
> - $\int ax^m + bx^n \, dx = \int ax^m \, dx + \int bx^n \, dx$ where a and b are constants.

- These same rules will now be applied for all real $n \neq -1$ and $m \neq -1$

- In general:

> - $\int k\, f(x)\, dx = k\int f(x) \, dx$ where k is a constant
> - $\int f(x) + g(x)\, dx = \int f(x)\, dx + \int g(x)\, dx$

Example 7.1

Without the use of a calculator, find the anti-derivative with respect to x, for:

(a) $\dfrac{x^4}{3}$ (b) $\dfrac{1}{x^4}$ (c) $\sqrt{x}$ (d) $\dfrac{1}{4\sqrt{x}}$.

Solution:

(a) The anti-derivative of $\dfrac{x^4}{3} = \dfrac{1}{3}\left(\dfrac{x^5}{5}\right) + C = \dfrac{x^5}{15} + C$.

(b) Rewrite $\dfrac{1}{x^4}$ as x^{-4} : $\Rightarrow$ anti-derivative of $\dfrac{1}{x^4} = \dfrac{x^{-3}}{-3} + C = -\dfrac{1}{3x^3} + C$.

(c) Rewrite $\sqrt{x}$ as $x^{\frac{1}{2}}$: $\Rightarrow$ anti-derivative of $x^{\frac{1}{2}} = \left(\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) + C = \dfrac{2}{3}x^{\frac{3}{2}} + C$.

(d) Rewrite $\dfrac{1}{4\sqrt{x}}$ as $\dfrac{1}{4}x^{-\frac{1}{2}}$: $\Rightarrow$ anti-derivative of $\dfrac{1}{4}x^{-\frac{1}{2}} = \dfrac{1}{4}\left(\dfrac{x^{\frac{1}{2}}}{\frac{1}{2}}\right) + C = \dfrac{1}{2}x^{\frac{1}{2}} + C$.

Example 7.2

Without the use of a calculator, integrate with respect to x:

(a) $\dfrac{x^2+x}{x^4}$ (b) $\left(1+\sqrt{x}\right)^2$.

Solution:

(a)
$$\int \frac{x^2+x}{x^4}\,dx = \int \frac{x^2}{x^4}+\frac{x}{x^4}\,dx = \int x^{-2}+x^{-3}\,dx$$
$$= \frac{x^{-1}}{-1}+\frac{x^{-2}}{-2}+C = -\frac{1}{x}-\frac{1}{2x^2}+C$$

(b)
$$\int\left(1+\sqrt{x}\right)^2 dx = \int 1+2\sqrt{x}+x\,dx$$
$$= x+2\left(\frac{x^{\frac{3}{2}}}{3/2}\right)+\frac{x^2}{2}+C$$
$$= x+\frac{4}{3}x^{\frac{3}{2}}+\frac{x^2}{2}+C$$

Example 7.3

Given that $f'(x)=x^2+\dfrac{1}{x^2}$, without the use of a calculator, find $f(x)$, given that $f(1)=1$.

Solution:

$f'(x)=x^2+\dfrac{1}{x^2} \quad \Rightarrow \quad f(x)=\dfrac{x^3}{3}-\dfrac{1}{x}+C$

But $f(1)=1$. $\quad \Rightarrow \quad 1=\dfrac{1}{3}-1+C$

$C=\dfrac{5}{3}$.

Therefore, $\quad f(x)=\dfrac{x^3}{3}-\dfrac{1}{x}+\dfrac{5}{3}$.

Example 7.4

Find $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)$ where $n \neq -1$. Hence, find the anti-derivatve of x^n.

Solution:

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{d}{dx}\left(\frac{1}{n+1}x^{n+1}\right)$$
$$= \frac{1}{n+1} \times (n+1)\, x^{n+1-1}$$
$$= x^n$$

Hence, anti-derivative of x^n is $\frac{x^{n+1}}{n+1} + C$.

Exercise 7.1

1. Integrate each of the following with respect to t:
   (a) $\frac{1}{t^2}$ (b) $\frac{1}{3t^3}$ (c) $\frac{1}{(3t)^3}$ (d) $\frac{-4}{t^4}$
   (e) $\frac{-5}{\pi^2}$ (f) $\left(\frac{2}{t}\right)^2$ (g) $\left(\frac{3}{2t}\right)^2$ (h) $\left(\frac{-2}{3t}\right)^3$
2. Anti-differentiate each of the following with respect to x:
   (a) $4\sqrt{x}$ (b) $\sqrt{2x}$ (c) $-5\sqrt[3]{x}$ (d) $\sqrt[3]{8x}$
   (e) $\frac{-4}{\sqrt{x}}$ (f) $\frac{1}{\sqrt[3]{27x}}$ (g) $\frac{-6}{\sqrt[5]{x}}$ (h) $\frac{3}{\sqrt[3]{8x^2}}$
3. Find the integral, with respect to t, for each of the following:
   (a) $\frac{1}{t^2} - \frac{1}{t^3}$ (b) $\frac{3}{t^4} - \frac{1}{\sqrt{t}}$ (c) $\frac{1}{2t^2} + \sqrt{t}$ (d) $\frac{t+1}{t^3}$
   (e) $\frac{1+t}{\sqrt{t}}$ (f) $\frac{\sqrt{t}-1}{t^2}$ (g) $2t + \frac{1+2t}{t^3}$ (h) $\frac{(t+4)^2}{t^4}$
4. Given that $f'(x) = (1+\sqrt{x})^2$, find $f(x)$ if $f(4) = 0$.
5. Given that $\frac{dy}{dx} = \left(1 - \frac{1}{x^2}\right)^2$, find y if $x = 1, y = 2$.
6. Given that $\frac{dV}{dt} = \left(\frac{2t^2+1}{t^2}\right)^2$, find V if $t = 1, V = 4$.

7.2 Anti-differentiation of $(ax+b)^n$ for $n \neq -1$

- Let $y = (ax+b)^{n+1}$ where $n \neq -1$.
  - Differentiating y with respect to x:

$$\frac{d}{dx}\left[(ax+b)^{n+1}\right] = (n+1)\times(ax+b)^{n+1-1}\times a$$
$$= a(n+1)(ax+b)^n$$

  - Since integration is the reverse of differentiation:

$$\int a(n+1)(ax+b)^n \; dx = (ax+b)^{n+1} + K$$
$$a(n+1)\int (ax+b)^n \; dx = (ax+b)^{n+1} + K$$
$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C.$$

- For $n \neq -1$, $\int (ax+b)^n \, dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + C$, where C is a constant.

- Note the pattern:

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

increase power by 1

coefficient of x term

new power

Example 7.5

Without the use of a calculator, find: (a) $\int (3-2t)^2 \; dt$ (b) $\int \frac{1}{\sqrt{1+4x}} \, dx$.

Solution:

(a)
$$\int (3-2t)^2 \; dt = \frac{(3-2t)^3}{3(-2)} + C = -\frac{(3-2t)^3}{6} + C$$

(b)
$$\int \frac{1}{\sqrt{1+4x}} \, dx = \int (1+4x)^{-1/2} \, dx$$
$$= \frac{(1+4x)^{1/2}}{\frac{1}{2}(4)} + C$$
$$= \tfrac{1}{2}\sqrt{1+4x} + C.$$

Example 7.6

Given that $\frac{dV}{dt} = \sqrt{4+3t}$, without the use of a calculator, find V given that $V(0) = 1$.

Solution:

$\frac{dV}{dt} = (4+3t)^{1/2} \quad \Rightarrow \quad V = \frac{(4+3t)^{3/2}}{\frac{3}{2}(3)} + C$

$\qquad = \frac{2}{9}(4+3t)^{3/2} + C$

$V(0) = 1$: $\qquad 1 = \frac{2}{9}(4)^{3/2} + C \quad \Rightarrow \quad C = -\frac{7}{9}$

Therefore $\qquad V = \frac{2}{9}(4+3t)^{3/2} - \frac{7}{9}$.

Exercise 7.2

1. Find the anti-derivative with respect to x, for each of the following:
   (a) $(2+x)^4$ (b) $(3+x)^5$ (c) $(1+2x)^3$ (d) $-(3+4x)^3$
   (e) $-\sqrt{(2+4x)}$ (f) $\sqrt[3]{1-3x}$ (g) $\sqrt[3]{(1+2x)^2}$ (h) $\sqrt{(2-5x)^3}$
   (i) $\frac{1}{\sqrt[3]{2+x}}$ (j) $\frac{-1}{\sqrt[3]{1-4x}}$ (k) $\frac{-1}{\sqrt{(1+x)^3}}$ (l) $\frac{1}{\sqrt[3]{(2+3x)^2}}$

2. Find:
   (a) $\int 4(x+3)^3\,dx$ (b) $\int \frac{(2x+5)^2}{4}\,dx$ (c) $\int \frac{-4}{(2x+3)^2}\,dx$ (d) $\int \frac{1}{2(1-5x)^4}\,dx$
   (e) $\int -4\sqrt{1-3x}\,dx$ (f) $\int \frac{\sqrt{2-5x}}{3}\,dx$ (g) $\int \frac{-2}{\sqrt{1-4x}}\,dx$ (h) $\int \frac{1}{4\sqrt{5+2x}}\,dx$

3. Integrate with respect to x:
   (a) $4x + 2(x^2-1)^2$ (b) $(x^2+2)^2/x^4 - (2x-1)^3$

4. For $\frac{dy}{dx} = 4(2+3x)^4$, find y given that when $x = 0$, $y = 1$.

5. For $\frac{dP}{dt} = \frac{-1}{(1+t)^2}$, find P given that when $t = 0$, $P = -1$.

6. For $f'(t) = \frac{2}{3\sqrt{1+3t}}$, find $f(t)$ given that $f(0) = 2$.

7. For $y' = (x/2+4)^3$, find y given that when $x = 0$, $y = 0$.

8. Given that $\frac{dV}{dt} = \frac{-2}{\sqrt{0.5t+4}}$, find V given that when $t = 0$, $V = 0$.

7.3 Anti-derivative of $f'(x)[f(x)]^n$

- Let $y = \dfrac{[f(x)]^{n+1}}{n+1}$ where $n \neq -1$.

  Using the Chain Rule: $\dfrac{d}{dx}\left(\dfrac{[f(x)]^{n+1}}{n+1}\right) = (n+1) \times \dfrac{[f(x)]^{n+1-1}}{n+1} \times f'(x)$

  $= f'(x)\,[f(x)]^n$.

- Hence:

$$\int f'(x).[f(x)]^n \, dx = \frac{[f(x)]^{n+1}}{n+1} + \text{C} \qquad n \neq -1$$

- Note the pattern:

  increase power by 1

$$\int f'(x).[f(x)]^n \, dx = \frac{[f(x)]^{n+1}}{n+1} + \text{C}$$

  Differentiate

  new power

Example 7.7

Without the use of a calculator, find: (a) $\int 5x(1+x^2)^4 \, dx$ (b) $\int \dfrac{10x}{\sqrt{1-3x^2}} \, dx$.

Solution:

(a) $\int 5x(1+x^2)^4 \, dx = 5\int x(1+x^2)^4 \, dx$

$= \dfrac{5}{2}\int 2x\,(1+x^2)^4 \, dx$

$= \dfrac{5}{2} \times \dfrac{(1+x^2)^5}{5} + C$

$= \dfrac{(1+x^2)^5}{2} + C.$

The integrand is now in the form $f'(x)\,[f(x)]^n$

(b) $\int \dfrac{10x}{\sqrt{1-3x^2}} \, dx = 10\int x(1-3x^2)^{-\frac{1}{2}} \, dx$

$= \dfrac{10}{-6}\int -6x\,(1-3x^2)^{-\frac{1}{2}} \, dx$

$= \dfrac{10}{-6} \times \dfrac{(1-3x^2)^{\frac{1}{2}}}{\frac{1}{2}} + \text{C}$

$= \dfrac{-10\,(1-3x^2)^{\frac{1}{2}}}{3} + \text{C}.$

The integrand is now in the form $f'(x)\,[f(x)]^n$

Exercise 7.3

1. Find:

(a) $\int 2x(5+x^2)^3\,dx$ (b) $\int x(1-x^2)^7\,dx$ (c) $\int 3x(x^2+4)^5\,dx$ (d) $\int 5x^2(3+2x^3)^4\,dx$

2. Evaluate:

(a) $\int \frac{5x}{(1+2x^2)^4}\,dx$ (b) $\int \frac{9x}{(3-4x^2)^3}\,dx$ (c) $\int \frac{x}{4(x^2-1)^3}\,dx$ (d) $\int \frac{5x^2}{7(2x^3+1)^3}\,dx$

3. Find:

(a) $\int -4x\sqrt{1+x^2}\,dx$ (b) $\int \frac{-3x\sqrt{1-x^2}}{4}\,dx$ (c) $\int \frac{3x}{\sqrt{1+2x^2}}\,dx$ (d) $\int \frac{2x}{3\sqrt{4x^2-5}}\,dx$

4. Find:

(a) $\int \left(5x^6\right)\sqrt[4]{1-x^7}\,dx$ (b) $\int \frac{3x^3}{\sqrt[5]{2x^4-7}}\,dx$ (c) $\int (1+x)(1+2x+x^2)^5\,dx$

(d) $\int \frac{x-x^3}{(3-2x^2+x^4)^5}\,dx$ (e) $\int \frac{1}{x^2}\sqrt{1+\frac{1}{x}}\,dx$ (f) $\int \frac{(1+\sqrt{x})^7}{\sqrt{x}}\,dx$

7.4 Anti-derivative of e^{mx} and e^{ax+b}

- Consider $y = e^{ax+b}$.

  Then, $\frac{d}{dx}(e^{ax+b}) = a e^{ax+b}$.

  Hence, $\int a e^{ax+b}\,dx = e^{ax+b} + K$

  $$a\int e^{ax+b}\,dx = e^{ax+b} + K$$

  $$\int e^{ax+b}\,dx = \frac{e^{ax+b}}{a} + C$$

- The rules for anti-differentiating exponential functions are listed below.

  - $\int e^{ax+b}\,dx = \frac{e^{ax+b}}{a} + C$
  - $\int e^{mx}\,dx = \frac{e^{mx}}{m} + C$
  - $\int e^{x}\,dx = e^{x} + C$

Example 7.8

Find: (a) $\int e^{0.01x}\,dx$ (b) $\int \frac{1}{e^{2x}}\,dx$ (c) $\int e^{3-2t}\,dt$

Solution:

(a) $$\int e^{0.01x}\,dx = \frac{e^{0.01x}}{0.01} + C = 100e^{0.01x} + C$$

(b) $$\int \frac{1}{e^{2x}}\,dx = \int e^{-2x}\,dx$$
$$= \frac{e^{-2x}}{-2} + C = -\tfrac{1}{2}e^{-2x} + C$$

(c) $$\int e^{3-2t}\,dt = \frac{e^{3-2t}}{-2} + C = -\tfrac{1}{2}e^{3-2t} + C$$

Exercise 7.4

1. Anti-differentiate with respect to x:
   (a) e^{2x} (b) $-e^{4x}$ (c) $e^{0.06x}$ (d) $-e^{0.05x}$
   (e) $10e^{0.02x}$ (f) $\frac{1}{10}e^{4x}$ (g) $\frac{4}{3}e^{0.4x}$ (h) $-\frac{4}{5}e^{-1.25x}$

2. Integrate with respect to t:
   (a) $1/e^{3t}$ (b) $-4/e^{0.01t}$ (c) $-1/(3e^{1.1t})$ (d) $3/(4e^{0.75t})$
   (e) $1/e^{-2t}$ (f) $3/e^{-t}$ (g) $-1/(2e^{-4t})$ (h) $4/(5e^{-0.05t})$

3. Find y given dy/dx:
   (a) e^{x+1} (b) e^{3x-4} (c) $2e^{2x+1}$ (d) $\frac{1}{2}e^{1+x}$
   (e) $1/e^{2x+1}$ (f) $2/e^{1-2x}$ (g) $3/(2e^{0.05x-1})$ (h) $-4/(5e^{1+0.05x})$

4. Find:
   (a) $\int 2x + 3e^{2x}\,dx$ (b) $\int 3x^2 + e^{-4x}\,dx$ (c) $\int \frac{1}{x^2} + e^{-3x+1}\,dx$
   (d) $\int -\frac{1}{4}e^{0.5x+1} + \frac{1}{x^2}\,dx$ (e) $\int (3x+1)^4 + \frac{1}{e^{2x-2}}\,dx$ (f) $\int \sqrt{e^x} + \frac{1}{(x+1)^2}\,dx$
   (g) $\int e^{-3x+1} + \frac{1}{\sqrt{1+2x}}\,dx$ (h) $\int e^x(e^x+1)^2\,dx$ (i) $\int e^{-2x}(1-e^{-2x})^4\,dx$

5. For $\frac{dV}{dt} = (1/e^t + e^t)^2$, find V, given that when $t = 0$, $V = 0$.

6. For $\frac{dP}{dt} = (1 + 2e^{-t})^2$, find P, given that when $t = 0$, $P = 1$.

7.5 Anti-derivative of $f'(x)\, e^{f(x)}$

- If $y = e^{f(x)}$, then $\dfrac{d}{dx} e^{f(x)} = f'(x)\, e^{f(x)}$.

  Since, anti-differentiation is the reverse of differentiation, then:

$$\boxed{\int f'(x) e^{f(x)}\, dx = e^{f(x)} + C}$$

- Note the pattern:

derivative

$$\int f'(x) e^{f(x)}\, dx = e^{f(x)} + C$$

carried through

Example 7.9

Find: (a) $\int (2x+3)\, e^{x^2+3x}\, dx$ (b) $\int x^2\, e^{x^3}\, dx$

Solution:

(a) $\int (2x+3)\, e^{x^2+3x}\, dx = e^{x^2+3x} + C.$

(b) $\int x^2\, e^{x^3}\, dx = \frac{1}{3}\int 3x^2\, e^{x^3}\, dx = \frac{1}{3} e^{x^3} + C.$

Exercise 7.5

1. Find the anti-derivative with respect to x for:

   (a) $2xe^{x^2}$ (b) $-3x^2 e^{-x^3}$ (c) $-2xe^{1-x^2}$ (d) $(2x+1)e^{x^2+x}$

   (e) $2xe^{-x^2}$ (f) $4x^3 e^{-x^4}$ (g) $4xe^{1-2x^2}$ (h) $(x^2+2x)e^{x^3+3x^2}$

2. Integrate with respect to x:

   (a) $\frac{1}{4} x e^{1+2x^2}$ (b) $-\frac{1}{2} x^2 e^{x^3+2}$ (c) $\frac{3}{2} x e^{x^2}$

   (d) $-\frac{4}{5} x^2 e^{x^3}$ (e) $\frac{2}{3}(x+1)e^{-2x-x^2}$ (f) $\dfrac{4x}{3e^{x^2+2}}$

3. Find: (a) $\int x e^{x^2}(1+3e^{x^2})^3\, dx$ (b) $\int \dfrac{xe^{2x^2}}{(1-e^{2x^2})^4}\, dx$ *(c) $\int \dfrac{11xe^{3x^2}}{\left[\sqrt{(1-e^{3x^2})}\right]^5}\, dx$

4. Verify that $3^{2x} = e^{2x\,ln(3)}$ where $ln\, 3 = \log_e 3$. Hence find the anti-derivative of 3^{2x}.

5. Verify that $2^{3x+1} = e^{(3x+1)ln(2)}$ where $ln\, 2 = \log_e 2$. Hence find the anti-derivative of 2^{3x+1}.

7.6 Standard Trigonometric Integrals

- Since $\frac{d}{dx}\sin x = \cos x$, $\int \cos x \, dx = \sin x + C$.
- Also $\frac{d}{dx}\cos x = -\sin x$, $\int \sin x \, dx = -\cos x + C$.
- In general:

$$\int \cos(ax+b)\, dx = \frac{\sin(ax+b)}{a} + C.$$

$$\int \sin(ax+b)\, dx = \frac{-\cos(ax+b)}{a} + C.$$

Example 7.10

Without the use of a calculator, find:

(a) $\int \cos(2x - \frac{\pi}{6})\, dx$ (b) $\int \sin(x)\cos(x)\, dx$ (c) $\int \sin(x)\cos^5(x)\, dx$

Solution:

(a) $$\int \cos(2x - \frac{\pi}{6})\, dx = \frac{\sin(2x - \frac{\pi}{6})}{2} + C$$

$$= \frac{1}{2}\sin(2x - \frac{\pi}{6}) + C$$

(b) $$\int \sin(x)\cos(x)\, dx = \int [\cos(x)][\sin(x)]^1\, dx$$

$$= \frac{\sin^2 x}{2} + C$$

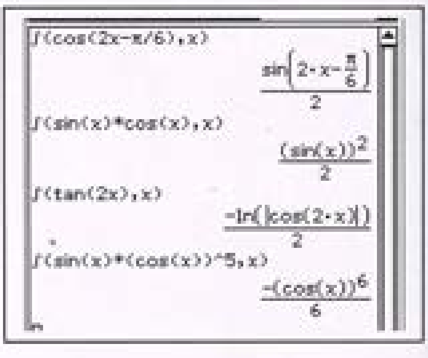
OR

$$\int \sin(x)\cos(x)\, dx = -\int [-\sin(x)][\cos(x)]^1\, dx$$

$$= -\frac{\cos^2 x}{2} + C$$

(c) $$\int \sin(x)\cos^5(x)\, dx = -\int [-\sin(x)][\cos(x)]^5\, dx$$

$$= -\frac{1}{6}\cos^6(x) + C$$



Exercise 7.6

1. Integrate each of the following with respect to the appropriate variable:
   (a) $\sin(2x)$ (b) $2\sin(t)$ (c) $\sin(1-2x)$
   (d) $\sin(\pi x)$ (e) $-\sin\left(\frac{4t}{3}\right)$ (f) $\frac{\sin 4x}{3}$
   (g) $3\sin\left(4t-\frac{\pi}{4}\right)$ (h) $\sin\left(\frac{1-2x}{3}\right)$ (i) $-\sin\left(\frac{\pi+2x}{6}\right)$

2. Integrate each of the following with respect to the appropriate variable:
   (a) $\cos(3t)$ (b) $-4\cos(x)$ (c) $\cos(1-\pi x)$
   (d) $\cos\left(\frac{-4t}{3}\right)$ (e) $\frac{\cos 5x}{2}$ (f) $\cos\left(1+\frac{x}{2}\right)$
   (g) $2\cos\left(4t+\frac{\pi}{3}\right)$ (h) $-\cos\left(\frac{1+3x}{4}\right)$ (i) $\cos\left(\frac{\pi}{4}\right)$

3. Integrate each of the following with respect to the appropriate variable:
   (a) $\cos(2x) - 4\sin(-x)$ (b) $e^{4x} + 3\sin(1-x)$
   (c) $2e^{-2x} + \cos(x/4)$ (d) $3x^2 + \sin(2x)$
   (e) $\sin(x)\cos^3(x)$ (f) $\cos(x)\sin^4(x)$
   (g) $\cos(2x)\sin^5(2x)$ (h) $1-2\sin^2(2x)$
   (i) $\sin(3x)\cos(2x) - \cos(3x)\sin(2x)$ (j) $\cos(4x)\cos(x) + \sin(4x)\sin(x)$

4. Differentiate $x\cos(x)$ with respect to x. Hence, find the anti-derivative of $x\sin(x)$.

5. Differentiate $\cos^2(x)$ with respect to x. Hence, find the anti-derivative of $\sin(x)\cos(x)$.

08 Definite Integrals

8.1 Area under a curve

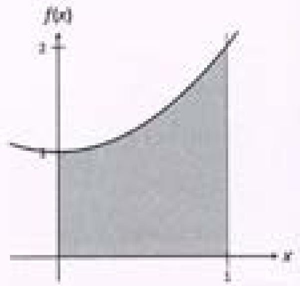
- In this section we will consider a calculus method for determining the area of the region trapped between the curve $y = f(x)$, the lines $x = a$ and $x = b$ and the x-axis.

Hands On Task 8.1

In this task, we will explore a procedure for determining the area of a region trapped between a curve (that is strictly increasing), the lines $x = a$ and $x = b$ and the x-axis.

Consider the region R trapped between the curve $y = x^2 + 1$, the lines $x = 0$, $x = 1$ and the x-axis (shown as the shaded region).

To determine the area of the shaded region R, we divide the shaded region into several rectangular strips of uniform width.



1. Divide the region into 5 rectangular strips of uniform width. These strips can be drawn (i) inscribed by the curve (Figure 8.1) or (ii) circumscribing the curve (Figure 8.2).

Figure 8.1

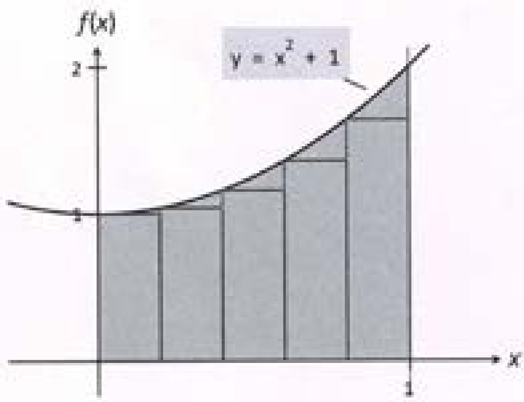
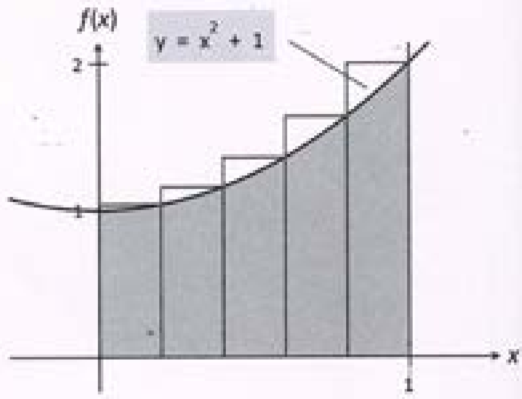


Figure 8·2



1. (a) Consider Figure 8.1. Let the width of a strip be represented by δx. Clearly, $\delta x = 0.2$. The height of the first strip $f(x_1) = f(0) = 1$. Complete the table below.

Strip Number i	Height of Strip $f(x_i)$	Width of Strip δx	Area of Strip $A_i = f(x_i) \times \delta x$
1	$f(0) = 1$	0.2	$1 \times 0.2 = 0.2$
2	$f(0.2)$		
3			
4			
5			

The total area of the inscribed strips $S_{inscribed} = A_1 + A_2 + A_3 + A_4 + A_5$.
The sum $A_1 + A_2 + A_3 + A_4 + A_5$ may be condensed using the summation notation as $\sum_{i=1}^{5} A_i$. Find $S_{inscribed}$.

(b) Consider Figure 8.2. The height of the first strip $f(x_1) = f(0.2) = 1 + 0.2^2 = 1.04$. Use the table below to find $S_{circumscribed}$, the total area of the circumscribing strips.

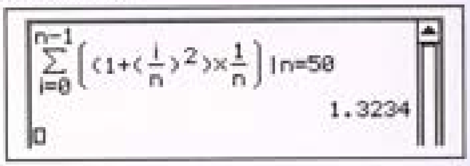
Strip Number i	Height of Strip $f(x_i)$	Width of Strip δx	Area of Strip $A_i = f(x_i) \times \delta x$
1	$f(0.2) = 1.04$	0.2	$1.04 \times 0.2 = 0.208$
2			
3			
4			
5			

(c) Between what two values should the area of region R lie.

2. (a) Verify that for n uniform rectangular strips $S_{inscribed} = \sum_{i=0}^{n-1} \left[1 + \left(\frac{i}{n} \right)^2 \right] \times \frac{1}{n}$.

(b) Verify that for n uniform rectangular strips $S_{circumscribed} = \sum_{i=1}^{n} \left[1 + \left(\frac{i}{n} \right)^2 \right] \times \frac{1}{n}$

3. The table below shows the values of $S_{inscribed}$ and $S_{circumscribed}$ for various values of n (number of strips). Use the results in question 2 and the summation function found in your CAS calculator to complete the table below. Comment on the values of $S_{inscribed}$ and $S_{circumscribed}$.



Number of Strips, n	$S_{inscribed}$ Total Area of Inscribed Strips	$S_{circumscribed}$ Total Area of Circumscribing Strips
50	1.323 40	1.343 40
100		
500		
1 000		
2 000		
5 000	1.333 233 34	1.333 433 34
10 000	1.333 283 335	1.333 383 335
15 000	1.333 300 001	1.333 366 67

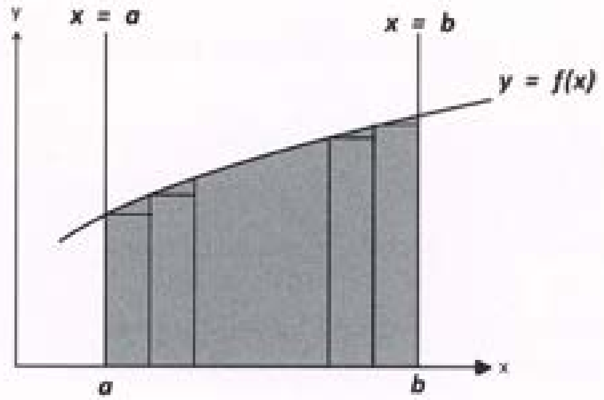
4. Use the table in question 3 to suggest a value for the area of region R, correct to 3 decimal places.

△ **Summary**

- As $n \to \infty$, $S_{inscribed} = S_{circumscribed}$.
- The area of the shaded region R = $\lim_{n\to\infty}\left[\sum_{i=0}^{i=n} f(x_i)\times\delta x\right]$.

8.2 The Riemann Integral

- In this section we will formalise the ideas explored in Hands On Task 8.1
- The accompanying diagram shows the graph of the <u>*non-negative*</u> curve $y = f(x)$. The area A of the region trapped between this curve, the x-axis and the lines $x = a$ and $x = b$ is to be calculated.



- We first approximate the area of the region. This is done by dividing it into n uniform rectangular strips each of width δx. The required area is then approximated by the total area of these strips.

Strip Number	Width of Strip	Height of Strip	Area of Strip
1	δx	$f(a)$	$f(a)\times\delta x$
2	δx	$f(a+\delta x)$	$f(a+\delta x)\times\delta x$
3	δx	$f(a+2\delta x)$	$f(a+2\delta x)\times\delta x$
n	δx	$f(b-\delta x)$	$f(b-\delta x)\times\delta x$

- Hence, $A \approx f(a)\times\delta x + f(a+\delta x)\times\delta x + f(a+2\delta x)\times\delta x + f(a+3\delta x)\times\delta x + \ldots + f(b-\delta x)\times\delta x$

  Using the summation notation: $A \approx \lim\limits_{\delta x\to 0} \sum\limits_{\substack{x=a \\ \text{increment } \delta x}}^{x=b-\delta x} f(x)\times\delta x$

- Clearly, as the number of strips n increases (as the width of the strip δx decreases) the approximation becomes more accurate.

  Hence: $A \approx \lim\limits_{\delta x\to 0} \sum\limits_{\substack{x=a \\ \text{increment } \delta x}}^{x=b-\delta x} f(x)\times\delta x = \lim\limits_{\delta x\to 0} \sum\limits_{\substack{x=a \\ \text{increment } \delta x}}^{x=b} f(x)\times\delta x$.

- The limit for the total area of the strips is *denoted* $\int_a^b f(x)\,dx$.

  That is, $\lim\limits_{\delta x \to 0} \sum\limits_{\substack{x=a \\ \textit{increment } \delta x}}^{x=b} f(x) \times \delta x = \int_a^b f(x)\,dx$.

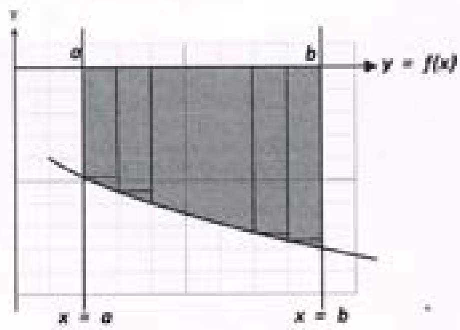
- $\int_a^b f(x)\,dx$ is called the ***Riemann integral***, named after Georg Friedrich Bernhard Riemann. "*a*" and "*b*" are respectively referred to as the ***lower and upper limits*** of the Riemann Integral.

- Note that the symbol $\int_a^b f(x)\,dx$ represents a numerical limit which refers to the area of the region trapped between the *non-negative* curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$.

- To evaluate $\int_a^b f(x)\,dx$, where $f(x) \geq 0$ for $a \leq x \leq b$, we need to evaluate the limit of the total area of the strips $\lim\limits_{\delta x \to 0} \sum\limits_{\substack{x=a \\ \textit{increment } \delta x}}^{x=b} f(x) \times \delta x$.

- If however, $f(x) \leq 0$ for $a \leq x \leq b$, that is the curve $y = f(x)$ is completely *below* the x-axis for $a \leq x \leq b$, $\lim\limits_{\delta x \to 0} \sum\limits_{\substack{x=a \\ \textit{increment } \delta x}}^{x=b} f(x) \times \delta x$ will be numerically negative.

  That is, if $f(x) \leq 0$ for $a \leq x \leq b$, $\int_a^b f(x)\,dx$ will be numerically negative.



- In this instance, the area of trapped region is given by

$$\lim_{\delta x \to 0} \sum_{\substack{x=a \\ \textit{increment } \delta x}}^{x=b} -f(x) \times \delta x = -\int_a^b f(x)\,dx.$$

8.3 $\int_a^b f(x)\,dx$ as Sum of Signed Areas

- In the previous section it was established that if:
  - $f(x) \geq 0$ for $a \leq x \leq b$, then the area of the region trapped between the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by

$$\lim_{\delta x \to 0} \sum_{\substack{x=a \\ \text{increment } \delta x}}^{x=b} f(x) \times \delta x \quad = \quad \int_a^b f(x)\,dx\,.$$

  - $f(x) \leq 0$ for $a \leq x \leq b$, then the area of the region trapped between the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by

$$\lim_{\delta x \to 0} \sum_{\substack{x=a \\ \text{increment } \delta x}}^{x=b} -f(x) \times \delta x \quad = \quad -\int_a^b f(x)\,dx\,.$$

- In this section, we will develop a geometrical interpretation for $\int_a^b f(x)\,dx$ where $f(x)$ is continuous in the interval $a \leq x \leq b$; in other words $f(x)$ is not necessarily non-negative in the interval $a \leq x \leq b$.

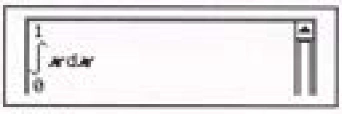
Hands On Task 8.2

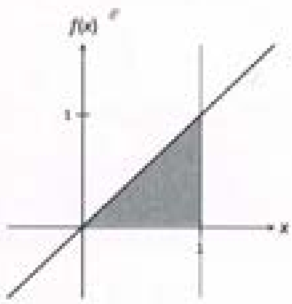
In this task, we will establish the concept of $\int_a^b f(x)\,dx$ as the sum of signed areas.

Locate the Riemann integral routine on your CAS/graphic calculator.

1. The sketch of $f(x) = x$ is given in the accompanying diagram. The region R is the region trapped between $f(x) = x$, and the lines $x = 0$, $x = 1$ and the x-axis.
   (a) Find A, the area of region R.
   (b) Use the Riemann integral routine to verify that

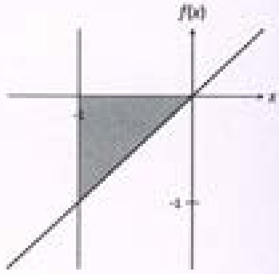
$$\int_0^1 x\,dx = \frac{1}{2}.$$





   (c) Compare your answers in (a) and (b).

2. The sketch of $f(x) = x$ is given in the accompanying diagram. The shaded region R is trapped between $f(x) = x$, and the lines $x = -1$, $x = 0$ and the x-axis.

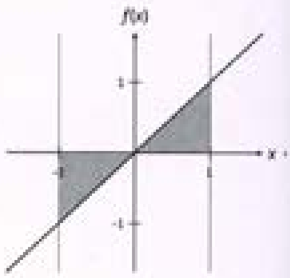


   (a) Find A, the area of region R.

   (b) Use the Riemann integral routine to find $\int_{-1}^{0} x \, dx$.

   (c) Verify that $\int_{-1}^{0} x \, dx = -A$.

3. The sketch of $f(x) = x$ is given in the accompanying diagram.



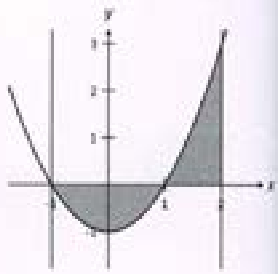
   (a) Find A_1, the area of region trapped between $f(x) = x$, and the lines $x = -1$, $x = 0$ and the x-axis.

   (b) Find A_2, the area of region trapped between $f(x) = x$, and the lines $x = 0$, $x = 1$ and the x-axis.

   (c) Use the Riemann integral routine to find $\int_{-1}^{1} x \, dx$.

   (d) Express $\int_{-1}^{1} x \, dx$ in terms of A_1 and A_2.

4. The sketch of $y = x^2 - 1$ is given in the accompanying diagram.



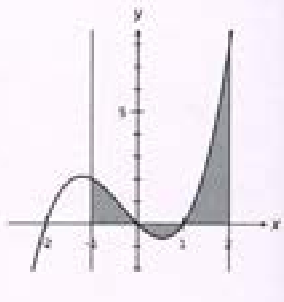
   (a) Find A_1, the area of region trapped between $y = x^2 - 1$ and the lines $x = -1$, $x = 1$ and the x-axis.

   (b) Find A_2, the area of region trapped between $y = x^2 - 1$ and the lines $x = 1$, $x = 2$ and the x-axis.

   (c) Use the Riemann integral routine to find $\int_{-1}^{2} x^2 - 1 \, dx$.

   (d) Express $\int_{-1}^{2} x^2 - 1 \, dx$ in terms of A_1 and A_2.

5. The sketch of $y = x(x - 1)(x + 2)$ is shown below.



   (a) Find A_1, the area of region trapped between this curve and the lines $x = -1$, $x = 0$ and the x-axis.

   (b) Find A_2, the area of region trapped between this curve and the lines $x = 0$, $x = 1$ and the x-axis.

   (c) Find A_3, the area of region trapped between this curve and the lines $x = 1$, $x = 2$ and the x-axis.

   (d) Express $\int_{-1}^{2} x(x-1)(x+2) \, dx$ in terms of A_1, A_2 and A_3.

△ Summary

- If $f(x) \geq 0$ for $a \leq x \leq b$ then

  $\int_a^b f(x)\,dx$ = Area of the region trapped between the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$.

- If $f(x) \leq 0$ for $a \leq x \leq b$ then

  $-\int_a^b f(x)\,dx$ = Area of the region trapped between the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$.

  - The area of the region trapped between the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by $-\int_a^b f(x)\,dx$.
  - On the other hand $\int_a^b f(x)\,dx$ has a value that is the *negative* of the area of the region trapped between the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$.

- If $y = f(x)$ is continuous in the interval $a \leq x \leq b$ then,

  $\int_a^b f(x)\,dx$ = Sum of *signed areas* of the regions trapped between the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$.

  - $\int_a^b f(x)\,dx$ will return a *positive* value if the sum of the areas of the regions under the curve $y = f(x)$ but above the x-axis for $a \leq x \leq b$ *exceeds* the sum of the areas of the regions below the x-axis.
  - In question 5 of Hands on Task 8.2, the area of the region trapped between the curve $y = x(x-1)(x+2)$ and the lines $x = 1$, $x = 2$ and the x-axis is

    Area = $A_1 + A_2 + A_3$

    $$= \int_{-1}^{0} x(x-1)(x+2)\,dx + \left[-\int_0^1 x(x-1)(x+2)\,dx\right] + \int_1^2 x(x-1)(x+2)\,dx.$$

  - But $\int_{-1}^{2} x(x-1)(x+2)\,dx$ has a numerical value of $A_1 - A_2 + A_3$.

- Hence, in general, the Riemann Integral $\int_a^b f(x)\,dx = \lim_{\delta x \to 0} \sum_{\substack{x=a \\ \text{increment } \delta x}}^{x=b} f(x) \times \delta x$

  is a numerical limit representing the sum of the signed areas of the regions trapped between the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$.

8.4 The Fundamental Theorem of Calculus I

Hands On Task 8.3

In this task, we will explore the Riemann Integral from another perspective to develop a more efficient technique for evaluating the Riemann Integral.

1. Consider $f(x) = x^2$.

   (a) Use the Riemann integral routine on your CAS calculator to evaluate $\int_0^1 f(x)\,dx$.

   (b) Find $F(x)$, the anti-derivative of $f(x)$. Hence, find $F(1) - F(0)$.

   (c) Comment on your answers in (a) and (b).

2. Consider $f(x) = 4x^3 + 3x^2$.

   (a) Use the Riemann integral routine on your CAS calculator to evaluate $\int_0^3 f(x)\,dx$.

   (b) Find $F(x)$, the anti-derivative of $f(x)$. Hence, find $F(3) - F(0)$.

   (c) Comment on your answers in (a) and (b).

3. Consider $f(x) = 2e^{2x}$.

   (a) Use the Riemann integral routine on your CAS calculator to evaluate $\int_{-1}^1 f(x)\,dx$.

   (b) Find $F(x)$, the anti-derivative of $f(x)$. Hence, find $F(1) - F(-1)$.

   (c) Comment on your answers in (a) and (b).

4. Consider $f(x) = 5x^4$.

   Suggest a procedure to evaluate $\int_{-1}^1 f(x)\,dx$ without using the calculator-based definite integral routine. Confirm (or refine) your procedure, by comparing the answer obtained using the suggested procedure with the answer obtained by using the calculator-based definite integral routine

5. Use the procedure you developed in Question 4, to evaluate:(a) $\int_0^1 e^x\,dx$ (b) $\int_1^2 \frac{1}{x^3}\,dx$.

Summary

- Given that $F(x)$ is an anti-derivative of $f(x)$ and $f(x)$ is continuous for $a \le x \le b$:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

8.4.1 The Riemann Integral and the Fundamental Theorem of Calculus

- The Riemann Integral $\int_a^b f(x)\,dx = \lim_{\delta x \to 0} \sum_{\substack{x=a \\ \text{increment } \delta x}}^{x=b} f(x) \times \delta x$.

  That is, to evaluate the integral, we need to evaluate the limit.

- The Fundamental Theorem of Calculus provides us with a more efficient procedure for evaluating the Riemann Integral $\int_a^b f(x)\,dx$.

- The Fundamental Theorem of Calculus establishes the link between definite integrals and anti-derivatives. Hence:

  $\int_a^b f(x)\,dx = F(b) - F(a)$ where $F(x)$ is ***an*** anti-derivative of $f(x)$.

  That is, the integral $\int_a^b f(x)\,dx$ may be evaluated using an anti-derivative of $f(x)$.

  The formal proof of this result is beyond the scope of this book.

- In the symbol for the Riemann Integral $\int_a^b f(x)\,dx$:

  - a is the lower limit of the interval $a \le x \le b$ and is usually called the "lower limit" of the definite integral
  - b is the upper limit of the interval $a \le x \le b$ and is usually called the "upper limit" of the definite integral.

- Note that $\int f(x)\,dx$ is used to denote the anti-derivative of $f(x)$ and is referred to as the *indefinite integral* of $f(x)$ with respect to x.

  The Riemann Integral $\int_a^b f(x)\,dx$ is referred to as the *definite integral* of $f(x)$ with respect to x between $x = a$ and $x = b$.

- The Fundamental Theorem of Calculus, allows us to evaluate a definite integral without using the method of numerical limits. However, not all definite integrals can be evaluated using the Fundamental Theorem of Calculus.

  - For example, $\int_0^1 e^{x^2}\,dx$ cannot be evaluated using the Fundamental Theorem of Calculus. To evaluate this definite integral, a numerical method is used.

8.5 Properties of the Definite Integral

- The properties of indefinite integrals may be extended to definite integrals as follows. For $f(x)$ and $g(x)$ continuous in the interval $a \leq x \leq b$:

  - $$\int_a^b k\, f(x)\, dx = k\int_a^b f(x)\, dx$$

  - $$\int_a^b f(x) + g(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

Example 8.1

Use the Fundamental Theorem of Calculus to evaluate $\int_0^1 x^2 + 1\, dx$.

Solution:

Let $f(x) = x^2 + 1$.

The anti-derivative of $f(x)$, $F(x) = \frac{1}{3}x^3 + x + C$.

Hence,
$$\int_0^1 x^2 + 1\, dx = F(1) - F(0)$$
$$= [\tfrac{4}{3} + C] - [C] = \tfrac{4}{3}$$

This solution is usually condensed into:

$$\int_0^1 x^2 + 1\, dx = \left[\tfrac{1}{3}x^3 + x\right]_0^1$$

An anti-derivative of f(x), F(x)

$$= \left[\tfrac{1}{3} + 1\right] - [0]$$

F(1)

F(0)

$$= \tfrac{4}{3}$$

Note:

- *Using the Fundamental Theorem of Calculus,* $\int_a^b f(x)\, dx = F(b) - F(a)$, *where F(x) is* ***an*** *anti-derivative of f(x). Clearly, it would be convenient to choose* $F(x) = \frac{1}{3}x^3 + x$, *with C = 0, as seen in the condensed version of the solution.*

Example 8.2

Without the use of a CAS/graphic calculator, evaluate each of the following.

(a) $\displaystyle\int_1^2 \frac{4}{t^2}\,dt$ (b) $\displaystyle\int_{-1}^2 e^{2x}\,dx$ (c) $\displaystyle\int_0^{\frac{\pi}{2}} \cos 2x\,dx$

Solution:

(a)

$$\begin{aligned}\int_1^2 \frac{4}{t^2}\,dt &= 4\int_1^2 t^{-2}\,dt\\ &= 4\left[\frac{-1}{t}\right]_1^2\\ &= 4\left[\frac{-1}{2}-\frac{-1}{1}\right]\\ &= 2\end{aligned}$$

(b)

$$\begin{aligned}\int_{-1}^2 e^{2x}\,dx &= \left[\tfrac{1}{2}e^{2x}\right]_{-1}^2\\ &= \tfrac{1}{2}\left[e^{2x}\right]_{-1}^2\\ &= \tfrac{1}{2}[e^4 - e^{-2}]\end{aligned}$$

(c)

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \cos 2x\,dx &= \left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}}\\ &= \frac{1}{2}\left(\sin\frac{\pi}{2}-\sin 0\right)\\ &= \frac{1}{2}.\end{aligned}$$

Exercise 8.1

1. Without the use of a CAS/graphic calculator, evaluate each of the following.

(a) $\displaystyle\int_0^1 x^2\,dx$ (b) $\displaystyle\int_{-1}^1 x+1\,dx$ (c) $\displaystyle\int_{-1}^1 2(x+1)^2\,dx$ (d) $\displaystyle\int_{-2}^2 4(1-x)^2\,dx$

(e) $\displaystyle\int_{-1}^0 \frac{1}{(x+2)^2}\,dx$ (f) $\displaystyle\int_0^2 \frac{1}{(1+2x)^3}\,dx$ (g) $\displaystyle\int_1^4 x+\frac{1}{\sqrt{x}}\,dx$ (h) $\displaystyle\int_0^1 e^x+1\,dx$

2. Without the use of a CAS/graphic calculator, find.

(a) $\displaystyle\int_0^1 \frac{2}{(1+2t)^3}\,dt$ (b) $\displaystyle\int_0^3 \sqrt{1+t}\,dt$ (c) $\displaystyle\int_0^1 \frac{1}{e^x}+x\,dx$ (d) $\displaystyle\int_{-1}^0 \frac{2}{e^{2x}}+e^x\,dx$

(e) $\displaystyle\int_{-1}^0 \sqrt{1+x}\,dx$ (f) $\displaystyle\int_{-1}^0 \frac{1}{\sqrt{1-x}}\,dx$ *(g) $\displaystyle\int_1^2 t e^{t^2}\,dt$ *(h) $\displaystyle\int_0^1 \frac{-3t}{(1+t^2)^2}\,dt$

3. By first evaluating the indefinite integral, find:

(a) $\displaystyle\int_0^1 x^2+2x+3\,dx$ (b) $\displaystyle\int_{-1}^1 (x+1)(x-2)\,dx$ (c) $\displaystyle\int_1^2 \frac{x+1}{x^3}\,dx$

(d) $\displaystyle\int_1^2 \frac{(x-1)^2}{x^4}\,dx$ (e) $\displaystyle\int_0^1 \frac{e^x+e^{2x}}{e^x}\,dx$ (f) $\displaystyle\int_0^1 (e^{-x}+1)^2\,dx$

(g) $\displaystyle\int_0^1 \frac{(e^{2t}+1)^2}{e^t}\,dt$ (h) $\displaystyle\int_0^1 \frac{-5t}{\sqrt{1+2t^2}}\,dt$ (i) $\displaystyle\int_0^1 \frac{-te^{t^2}}{\sqrt{1+e^{t^2}}}\,dt$

4. Find the indefinite integral first, then evaluate the following integrals leaving your answers in exact form.

(a) $\displaystyle\int_{-\pi}^{\pi/3} \sin(2x)\,dx$ (b) $\displaystyle\int_{-\pi}^{\pi} \cos(x/2)\,dx$ (c) $\displaystyle\int_{-\pi}^{-\pi/2} x+\sin(x+\pi)\,dx$

(d) $\displaystyle\int_0^1 \cos(\pi x)\,dx$ (e) $\displaystyle\int_0^2 \cos(\pi/2-\pi x)\,dx$ (f) $\displaystyle\int_0^4 \sin(\pi x/3)+e^x\,dx$

5. Given that $\dfrac{d}{dx}[\sin(x)-x\cos(x)]=x\sin(x)$, determine exactly $\displaystyle\int_0^{\pi/2} x\sin(x)\,dx$.

6. Given that $\dfrac{d}{dx}[e^x(\sin(x)+\cos(x))]=[2e^x\cos(x)]$, determine exactly $\displaystyle\int_{-\pi/2}^{\pi/2} 2e^x\cos(x)\,dx$.

7. Given that $\dfrac{d}{dx}[x\ln(x)-x]=\ln(x)$, determine exactly $\displaystyle\int_1^e -\ln(x)\,dx$.

8. Given that $\dfrac{d}{dx}[xe^x-e^x]=xe^x$, determine exactly $\displaystyle\int_0^1 [xe^x+x^2]\,dx$.

9. Differentiate $e^{-x}[\sin(x)+\cos(x)]$ with respect to x. Hence, or otherwise, find the exact value of $\displaystyle\int_0^{\pi} [e^{-x}\sin(x)+\cos(x)]\,dx$.

8.6 Fundamental Theorem of Calculus II

- In Section 8.3, the Fundamental Theorem of Calculus was described as follows.
  - Given that $F(x)$ is an anti-derivative of $f(x)$ and $f(x)$ is continuous in the interval $a \le x \le b$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.
  - This part is best associated with the evaluation of definite integrals.

- The Fundamental Theorem of Calculus may also be described as follows.
  - If $f(t)$ is continuous in the interval $a \le t \le b$, and if $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$.
  - That is, if $f(t)$ is continuous in the interval $a \le t \le b$, then

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x).}$$

  - This part establishes the connection between definite integrals and anti-derivatives.

- In general,
  - If $f(t)$ is continuous in the interval $a \le t \le b$, and if $F(x) = \int_a^{g(x)} f(t)\,dt$, then by the chain Rule for differentiation $F'(x) = f[g(x)] \times g'(x)$.
  - That is, if $f(t)$ is continuous in the interval $a \le t \le b$,

then $$\boxed{\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f[g(x)] \times g'(x).}$$

- This part of the Fundamental Theorem of Calculus will be explored in greater detail in Section 9.4.1.

Example 8.3

Find the derivative of $F(x)$ given that (a) $F(x) = \int_1^x e^{t^2}\ dt$ (b) $F(x) = \int_1^{x^2+1} 2t+1\ dt$

Solution:

(a) Using the Fundamental Theorem of Calculus: $F'(x) = e^{x^2}$.

(b) Let $f(t) = 2t + 1$, $g(x) = x^2 + 1$
Using the Fundamental Theorem of Calculus:

$$\begin{aligned} F'(x) &= f[g(x)] \times g'(x) \\ &= [2(x^2 + 1) + 1] \times 2x \\ &= 2x(2x^2 + 3). \end{aligned}$$

Example 8.4

Find: (a) $\frac{d}{dx}\int_0^x 4t+1\ dt$ (b) $\frac{d}{dx}\int_0^{e^{2x}} 4t+1\ dt$

Solution:

(a) Using the Fundamental Theorem of Calculus: $\frac{d}{dx}\int_0^x 4t+1\ dt = 4x + 1$

(b) Using the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_0^{e^{2x}} 4t+1\ dt = [4\,e^{2x} + 1] \times 2\,e^{2x}\ .$$

Example 8.5

Find $\int_0^1 \frac{d}{dx}\left[\frac{1-\sqrt{x}}{1+\sqrt{x}}\right] dx$.

Solution:

$$\begin{aligned} \int_0^1 \frac{d}{dx}\left[\frac{1-\sqrt{x}}{1+\sqrt{x}}\right] dx &= \left[\frac{1-\sqrt{x}}{1+\sqrt{x}}\right]_0^1 \\ &= \left[\frac{1-\sqrt{1}}{1+\sqrt{1}}\right] - \left[\frac{1-\sqrt{0}}{1+\sqrt{0}}\right] \\ &= -1 \end{aligned}$$

Exercise 8.2

1. Find the derivative of each of the following integrals.

(a) $\int_1^x 4t + t^2 \; dt$ (b) $\int_1^x \sqrt{1+t^2} \; dt$ (c) $\int_1^x 4te^{t^2} \; dt$ (d) $\int_x^1 \frac{1}{\sqrt{u}} \; du$

(e) $\int_1^{2x} t^3 + t^2 \; dt$ (f) $\int_{t^2}^0 \frac{1-u}{1+u} \; du$ (g) $\int_1^{\sqrt{x}} \frac{e^{t^2}}{4t} \; dt$ (h) $\int_x^{x+5} \sqrt{1+e^u} \; du$

(i) $\int_1^x u\cos(u) \; du$ (j) $\int_1^{\sin x} u\sin(u) \; du$ (k) $\int_1^{\sin(u)} \frac{e^{t^2}}{4t} \; dt$ (l) $\int_1^{2x} \sqrt{1+e^u} \; du$

2. Find the following:

(a) $\int_1^x \frac{d}{dt}[t^2 + 4t] \; dt$ (b) $\int_{x^2}^{\pi} \frac{d}{du}[u^2 + 4u] \; du$ (c) $\frac{d}{dx}\int_1^x t^2 e^t \; dt$

(d) $\frac{d}{dx}\int_{\pi}^{e^x} u^2 + \sqrt{u} \; du$ (e) $\frac{d}{dx}\int_0^{2x} \sqrt{1+t^2} \; dt$ (f) $\frac{d}{dt}\int_2^{t^2} \frac{1-u}{1+u^2} \; du$

(g) $\int_1^{e^{2x}} \frac{d}{dt}[\frac{e^t}{t^2}] \; dt$ (h) $\int_{\frac{1}{x}}^1 \frac{d}{du}[u\sqrt{1+u^2}\,] \; du$ (i) $\int_x^{x+2} \frac{d}{du}[\sqrt{1+u^3}\,] \; du$

(j) $\int_{\pi}^x \frac{d}{dt}[\sin^2(t)] \; dt$ (k) $\int_{\pi}^{x^2} \frac{d}{du}[\tan(u)] \; du$ (l) $\frac{d}{dx}\int_{\pi}^{e^x} u^2 \sin(u) \; du$

3. Find the value of F´(1), given that $F(x) = \int_0^x \frac{t}{t^2+4} \; dt$.

4. Find the value of F´(π/2), given that $F(x) = \int_0^x \sin(t) \; dt$.

5. Find the value of F´(0), given that $F(x) = \int_1^x e^{t^2} \; dt$.

6. Find the x-coordinate of the stationary point of the curve:

(a) $y = \int_1^{2x} t^2 + 4t \; dt$ (b) $y = \int_1^{x+1} t^2(t-2) \; dt$

(c) $y = \int_1^{x^2-4} \sqrt{t}\, e^t \; dt$ (d) $y = \int_1^{x-1} \frac{1-u}{\sqrt{1+u}} \; du$.

8.7 More Properties of Definite Integrals

- For $f(x)$ and $g(x)$ each continuous in the interval $a \le x \le b$:

  - $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

  - $$\int_a^b f(x)\,dx = \int_a^m f(x)\,dx + \int_m^b f(x)\,dx \quad \text{where } a \le m \le b.$$

Example 8.6

Given that $f(x)$ is continuous everywhere and that $\int_{-4}^{2} f(x)\,dx = 10$ and $\int_{-1}^{2} f(x)\,dx = 6$, find:

(a) $\int_{2}^{-4} f(x)\,dx$ (b) $\int_{-4}^{2} 3f(x)\,dx$ (c) $\int_{-4}^{-1} f(x)\,dx$.

Solution:

(a) $$\int_{2}^{-4} f(x)\,dx = -\int_{-4}^{2} f(x)\,dx = -10$$

(b) $$\int_{-4}^{2} 3f(x)\,dx = 3\int_{-4}^{2} f(x)\,dx$$
$$= 3 \times 10 = 30$$

(c) $$\int_{-4}^{2} f(x)\,dx = \int_{-4}^{-1} f(x)\,dx + \int_{-1}^{2} f(x)\,dx$$
$$10 = \int_{-4}^{-1} f(x)\,dx + 6$$
$$\Rightarrow \int_{-4}^{-1} f(x)\,dx = 4$$

Example 8.7

Given that $f(x)$ is continuous everywhere and that $\int_{-4}^{6} f(x)\,dx = 12$, find:

(a) $\int_{-4}^{6} 2x - 2f(x)\,dx$ (b) $\int_{-2}^{8} 3f(x-2)\,dx$ (c) $\int_{-2}^{3} f(2x)\,dx$.

Solution:

(a)

$$\begin{aligned}\int_{-4}^{6} 2x - 2f(x)\,dx &= \int_{-4}^{6} 2x\,dx + \int_{-4}^{6} -2f(x)\,dx \\ &= \left[x^2\right]_{-4}^{6} - 2\int_{-4}^{6} f(x)\,dx \\ &= (36-16) - 2\times 12 \\ &= -4\end{aligned}$$

(b)

$$\begin{aligned}\int_{-2}^{8} 3f(x-2)\,dx &= 3\int_{-2}^{8} f(x-2)\,dx \\ &= 3\int_{-4}^{6} f(x)\,dx \\ &= 3\times 12 \\ &= 36\end{aligned}$$

(c)

$$\begin{aligned}\int_{-2}^{3} f(2x)\,dx &= \frac{1}{2}\times\int_{-4}^{6} f(x)\,dx \\ &= \frac{1}{2}\times 12 \\ &= 6\end{aligned}$$

Notes:

- *In part (b),* $\int_{-2}^{8} f(x-2)\,dx \xrightarrow{\text{Translate 2 units Left}} \int_{-4}^{6} f(x)\,dx$,

  the sum of the signed areas remain unchanged. Hence, $\int_{-2}^{8} f(x-2)\,dx = \int_{-4}^{6} f(x)\,dx$.

- *In part (c),* $\int_{-2}^{3} f(2x)\,dx \xrightarrow{\text{Dilate Horizontally factor 2}} \int_{-4}^{6} f(x)\,dx$

  the sum of the signed areas is doubled. Hence, $\int_{-2}^{3} f(2x)\,dx = \frac{1}{2}\times\int_{-4}^{6} f(x)\,dx$.

Exercise 8.3

1. Given that $f(x)$ is continuous everywhere and $\int_2^6 f(x)\,dx = 5$ and $\int_2^4 f(x)\,dx = 2$, find:

(a) $\int_2^6 -2f(x)\,dx$ (b) $\int_2^6 [f(x)+1]\,dx$ (c) $\int_2^6 [x+f(x)]\,dx$

(d) $\int_6^2 f(x)\,dx$ (e) $\int_6^2 \frac{f(x)}{2}\,dx$ (f) $\int_4^6 f(x)\,dx$

2. Given that $f(x)$ is continuous everywhere and $\int_{-4}^6 f(x)\,dx = 7$ and $\int_3^6 f(x)\,dx = 3$, find:

(a) $\int_{-4}^6 2f(x)\,dx$ (b) $\int_{-4}^6 [f(x)-x]\,dx$ (c) $\int_6^3 [x+f(x)]\,dx$

(d) $\int_6^{-4} [-3f(x)+4]\,dx$ (e) $\int_{-4}^3 \frac{f(x)}{2}\,dx$ (f) $\int_3^{-4} [1-2f(x)]\,dx$

3. Given that $f(x)$ and $g(x)$ are each continuous everywhere and
$\int_1^4 f(x)\,dx = 3$, $\int_4^7 f(x)\,dx = -2$ and $\int_1^7 g(x)\,dx = 4$, find:

(a) $\int_1^7 f(x)\,dx$ (b) $\int_7^1 \frac{-f(x)}{3}\,dx$ (c) $\int_1^7 [x^2+g(x)]\,dx$

(d) $\int_1^7 [f(x)+g(x)]\,dx$ (e) $\int_1^7 \frac{f(x)-g(x)}{2}\,dx$ (f) $\int_1^7 [3g(x)-2f(x)]\,dx$

4. Given that $f(x)$ and $g(x)$ are each continuous everywhere and that
$\int_{-7}^{-4} f(x)\,dx = -3$, $\int_{-4}^{-1} f(x)\,dx = 1$ and $\int_{-4}^{-1} g(x)\,dx = -2$, find:

(a) $\int_{-7}^{-1} x-\frac{f(x)}{2}\,dx$ (b) $\int_{-1}^{-4} g(x)+2f(x)\,dx$ (c) $\int_{-1}^{-4} e^x+g(x)+f(x)\,dx$

(d) $\int_{-1}^{-4} \frac{f(x)}{2}-\frac{3g(x)}{4}\,dx$ (e) $\int_{-4}^{-1} \pi f(x)-\frac{g(x)}{\pi}\,dx$ (f) $\int_{-1}^{-4} \frac{f(x)}{4}-\pi g(x)\,dx$

5. Given that $f(x)$ is continuous everywhere and that $\int_{-2}^{8} f(x)\,dx = 20$, find:

(a) $\int_{-2}^{8} 1-2f(x)\,dx$ (b) $\int_{-3}^{7} 2f(x+1)\,dx$ (c) $\int_{-4}^{16} f(0.5x)\,dx$

(d) $\int_{2}^{-8} f(-x)\,dx$ (e) $\int_{0}^{5} f(2x-2)\,dx$ (f) $\int_{-7}^{3} f(1-x)\,dx$

6. Given that $f(x)$ is continuous everywhere and that $\int_{4}^{10} f(x)\,dx = -10$, find:

(a) $\int_{4}^{10} 2x+f(x)\,dx$ (b) $\int_{5}^{11} f(x-1)\,dx$ (c) $\int_{1}^{3} f(3x+1)\,dx$

(d) $\int_{-10}^{-4} -f(-x)\,dx$ (e) $\int_{10}^{22} f\left(\frac{x-2}{2}\right)dx$ (f) $\int_{-3}^{-9} 2f(1-x)\,dx$

7. Given that $f(x)$ is continuous everywhere and that $\int_{-3}^{5} f(x+2)\,dx = 4$, find:

(a) $\int_{-3}^{5} -4x+f(x+2)\,dx$ (b) $\int_{-1}^{7} 1+f(x)\,dx$ (c) $\int_{-5}^{3} f(-x+2)\,dx$

(d) $\int_{-7}^{1} -f(-x)\,dx$ (e) $\int_{-2}^{6} f(x+1)\,dx$ (f) $\int_{1}^{9} 2f(x-2)\,dx$

8. Given that $f(x)$ is continuous everywhere and that $\int_{-2}^{6} f(2x)\,dx = -5$, find:

(a) $\int_{-2}^{6} f(2x)-2x\,dx$ (b) $\int_{-4}^{12} f(x)\,dx$ (c) $\int_{-3}^{5} f(2x+2)\,dx$

(d) $\int_{-6}^{2} \frac{f(-2x)}{2}\,dx$ (e) $\int_{-3}^{13} f(x-1)\,dx$ (f) $\int_{-4}^{4} 2f(4-2x)\,dx$

09 Applications of Integration

9.1 Gradient Function

- In Section 5.1, we noted that if $y = f(x)$ represents the equation of a curve, then $\frac{dy}{dx} = f'(x)$ represents the gradient function of the curve.
- In this section, we will reverse the process; given the gradient function of a curve, we will be required to determine the actual equation of the curve.

Example 9.1

The gradient function of a curve is given by $\frac{dy}{dx} = 1 - \cos x$.

Find the equation of the curve if the curve passes through the point (0, 1).

Solution:

$$\frac{dy}{dx} = 1 - \cos x \qquad \Rightarrow \quad y = \int 1 - \cos x \, dx$$

$$= x - \sin x + C$$

When $x = 0, y = 1$. $\Rightarrow$ $C = 1$

The equation of the curve is:

$$y = x - \sin x + 1.$$

Notes:

- *Curves with equation of the form $y = x - \sin x + C$ share the same gradient function $\frac{dy}{dx} = 1 - \cos x$.*
  That is, the gradient function $\frac{dy}{dx} = 1 - \cos x$ is common to the group of curves with equation $y = x - \sin x + C$.
- *In general, $y = \int \frac{dy}{dx}\, dx$ gives a family of curves. Individual curves are obtained only when a point of passage is known.*

Exercise 9.1

1. Find the equation of the curve with gradient function $\frac{dy}{dx}$ and passing through the point:

   (a) $\frac{dy}{dx} = 6x^2 + 2x - 5$, (1, 0)

   (b) $\frac{dy}{dx} = e^x + x + 1$, (0, 2)

   (c) $\frac{dy}{dx} = e^{-x} - 2x$, (0, −1)

   (d) $\frac{dy}{dx} = \frac{1}{\sqrt{x+2}}$, (−1, 0)

   (e) $\frac{dy}{dx} = \frac{x}{\sqrt{x^2+1}}$, (0, 0)

   (f) $\frac{dy}{dx} = 2 + \sin x$, (0, 1)

2. Find the equation of the curve with gradient function $\frac{dy}{dx}$ and passing through the point:

   (a) $\frac{dy}{dx} = \frac{1}{x^2}$, (1, 2)

   (b) $\frac{dy}{dx} = \frac{1}{(x+1)^2}$, (0, 1)

   (c) $\frac{dy}{dx} = \sqrt{(x+1)}$, (0, 1)

   (d) $\frac{dy}{dx} = \frac{x}{(1+x^2)^2}$, (0, 1)

   (e) $\frac{dy}{dx} = \frac{1}{\sqrt{x}} + \frac{1}{e^x}$, (0, 2)

   (f) $\frac{dy}{dx} = \sin(\pi x)$, (1/2, π)

3. The curve $y = f(x)$, has gradient function given by $\frac{dy}{dx} = x^2 + b$. The curve has turning points, at $x = -1$ and $x = 1$. Find the equation of the curve given that it passes through the origin.

4. The curve $y = f(x)$, has gradient function given by $f(x) = x^2 + bx + c$. The curve has two turning points, at $x = -2$ and $x = 0$. Find the equation of the curve given that it passes through the point (1, 1).

5. The tangent to the curve $y = f(x)$ at the point (0, 1) has equation $y = x + 1$. The gradient function of the curve is given by $\frac{dy}{dx} = ax + b$. Find the equation of the curve given that it also passes through the point (1, 3).

6. The tangent to the curve $y = f(x)$ at the point (0, −1) has equation $y = 2x - 1$. The gradient function of the curve is given by $\frac{dy}{dx} = ae^x + b$. Find the equation of the curve given that it also passes through the point (1, 0).

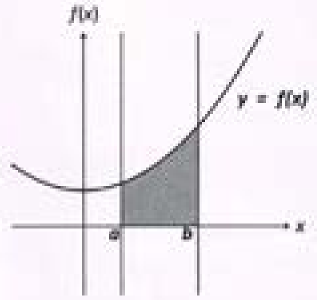
7. The curve $y = f(x)$ is such that $\frac{d^2y}{dx^2} = x + a$. Given that the curve has an inflection point at (1, −1/3), and turning points at (0, 0) and (2, −2/3), find the equation of the curve.

9.2 Area under a Curve

9.2.1 Area trapped between curve and the x-axis

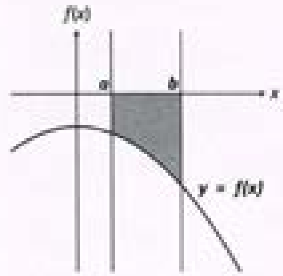
- $\int_a^b f(x)\,dx$ represents the area of the region trapped between $y = f(x)$, the lines $x = a$, $x = b$ and the x-axis *only* if $f(x) \geq 0$ within the interval $a \leq x \leq b$; that is, $y = f(x)$ is completely above the x-axis in this interval.
- If $f(x) \leq 0$ within the interval $a \leq x \leq b$, $y = f(x)$ is completely below the x-axis in this interval, then the area of the region trapped between $y = f(x)$, the lines $x = a$, $x = b$ and the x-axis, is represented by $-\int_a^b f(x)\,dx$. This is because, for $f(x) \leq 0$ in the interval $a \leq x \leq b$, $\int_a^b f(x)\,dx$ is numerically negative.
- If, the region has parts on both sides of the x-axis, then the sub-parts must be calculated separately.
- Hence, the following results.

  - Area of Shaded Region $= \int_a^b f(x)\,dx$.



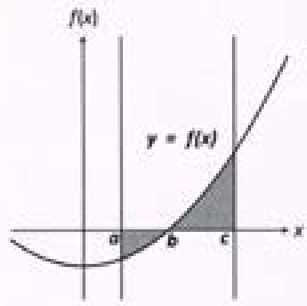
  - Area of Shaded Region $= -\int_a^b f(x)\,dx$

    or $\left|\int_a^b f(x)\,dx\right|$



  - Area of Shaded Region

    $= -\int_a^b f(x)\,dx + \int_b^c f(x)\,dx$



- To calculate the area of the region trapped between $y = f(x)$, the lines $x = a$, $x = b$, and the x-axis:
    - Make a quick sketch of $y = f(x)$, either by hand or using a CAS/graphic calculator.
    - Identify the location of the trapped region and the points where $y = f(x)$ crosses the x-axis (if applicable).
    - If the trapped region occupies both sides of the x-axis, work out each sub-area separately. The required area is the sum of the sub-areas.
- Where the use of CAS/graphic calculator is permitted, the area of the region trapped between $y = f(x)$, the lines $x = a$, $x = b$, and the x-axis, may be calculated using the formula:

$$\text{Area} = \int_a^b |f(x)|\, dx .$$

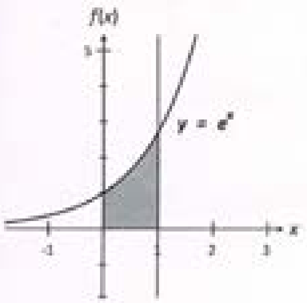
This formula may be used without the need to partition the trapped region into a region above and a region below the x-axis.

- Note that $\int_a^b f(x)\, dx$ gives the *sum of the signed areas* whereas $\int_a^b |f(x)|\, dx$ gives the area of the regions trapped between the curve $y = f(x)$, the lines $x = a$, $x = b$, and the x-axis

Example 9.2

Without the use of a CAS/graphic calculator, find the area of the shaded region.

(a)



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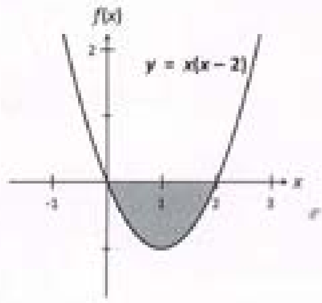


Solution:

(a) Area of Shaded Region $= \int_0^1 e^x\, dx = \left[e^x\right]_0^1 = e^1 - e^0 = e - 1$

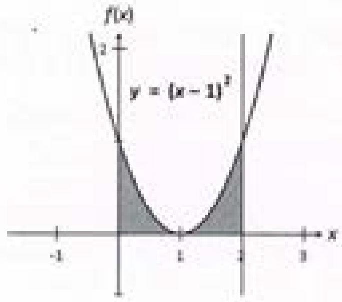
(b) Area of Shaded Region $= -\int_0^2 x^2 - 2x\, dx = \left[\frac{x^3}{3} - x^2\right]_0^2$

$$= -[(\tfrac{8}{3} - 4) - (0)] = \tfrac{4}{3}.$$

Example 9.3

Without the use of a CAS/graphic calculator, find the area of the shaded region.

(a)



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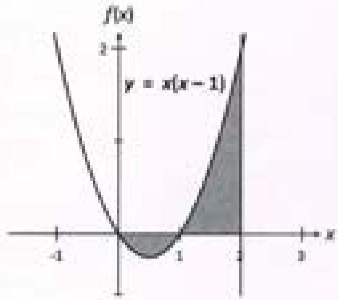


Solution:

(a) Area of Shaded Region $= \int_0^2 (x-1)^2\, dx = \left[\frac{(x-1)^3}{3}\right]_0^2 = \frac{1}{3} - \frac{-1}{3} = \frac{2}{3}$.

(b) Area of Shaded Region $= -\int_0^1 x^2 - x\, dx + \int_1^2 x^2 - x\, dx$

$$= -\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2$$

$$= -\left[\frac{1}{3} - \frac{1}{2}\right] + \left[\left(\frac{8}{3} - \frac{4}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)\right]$$

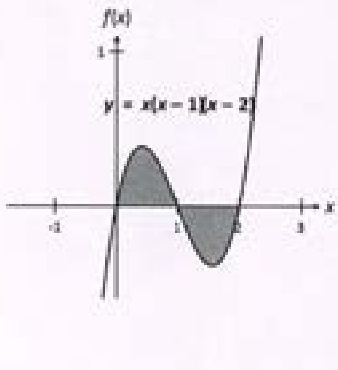
$$= \frac{1}{6} + \frac{5}{6} = 1.$$

Example 9.4

Find the area of the region tapped by the curve $y = x(x-1)(x-2)$ and the x-axis.

Solution:

The sketch of $y = x(x-1)(x-2)$ is given in the accompanying diagram. The region trapped by the curve $y = x(x-1)(x-2)$ and the x-axis is shown as the shaded region.



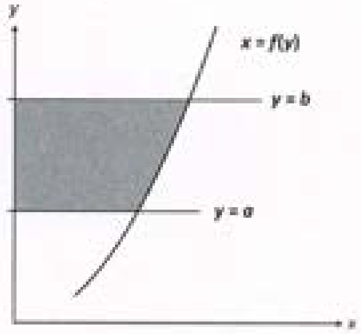
The roots of $y = x(x-1)(x-2)$ are $x = 0, 1, 2$.

Therefore Area of Shaded Region

$$= \int_0^2 |x(x-1)(x-2)|\, dx = \frac{1}{2}.$$

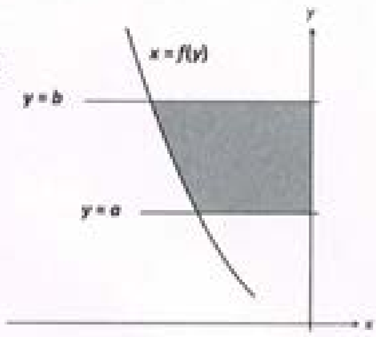
9.2.2 With the y-axis as a Boundary

- Using the technique demonstrated in Section 8.1, but with horizontal strips, the area of the region trapped between the curve $x = f(y)$, the y-axis, to the right of the y-axis, and the lines $y = a$ and $y = b$, is given by $A = \int_a^b f(y)\, dy$.



- Note that the equation of the curve is written with y as the independent variable, $x = f(y)$. Also, note that y is the subject of the integration, and the region is to the right of the y-axis.

- Similarly, the area of the region trapped between the curve $x = f(y)$, the y-axis, and to the left of the y-axis, between the lines $y = a$ and $y = b$ is given by
$$A = -\int_a^b f(y)\, dy = \left| \int_a^b f(y)\, dy \right|.$$



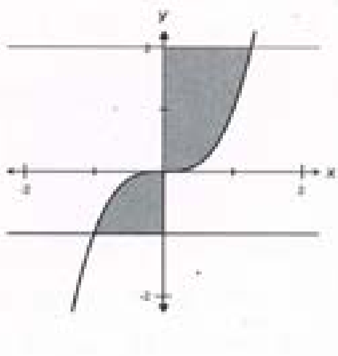
- Combining the two earlier results:

> The area of the region trapped between the curve $x = f(y)$, the y-axis, and the lines $y = a$ and $y = b$ is given by $A = \int_a^b |f(y)|\, dy$.

Example 9.5

Determine the area of the region trapped between the curve $y = x^3$, the y-axis and the lines $y = 1$ and $y = 2$.

Solution:



$y = x^3 \Rightarrow x = y^{\frac{1}{3}}$

Area of trapped region is given by:

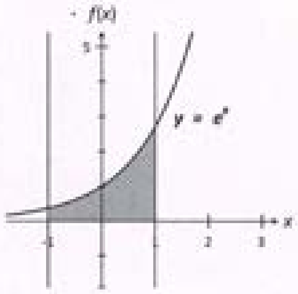
$$A = -\int_{-1}^{0} y^{\frac{1}{3}}\, dy + \int_{0}^{2} y^{\frac{1}{3}}\, dy$$

$$= -\left[\frac{3y^{\frac{4}{3}}}{4}\right]_{-1}^{0} + \left[\frac{3y^{\frac{4}{3}}}{4}\right]_{0}^{2} = 2.6399$$

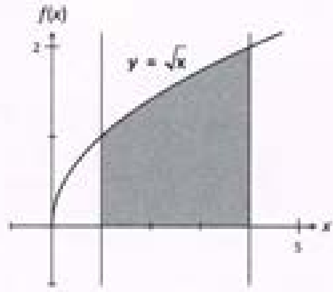
Exercise 9.2

1. Determine the area of the shaded region:

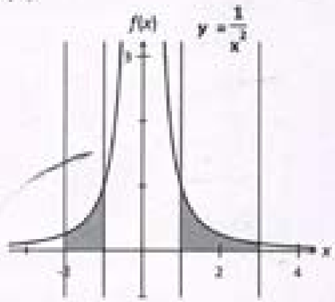
(a)



(b)



(c)



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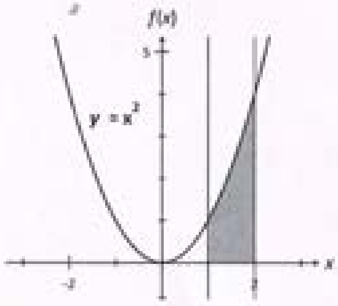

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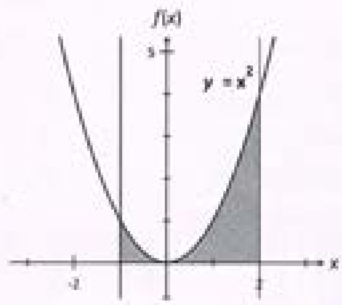

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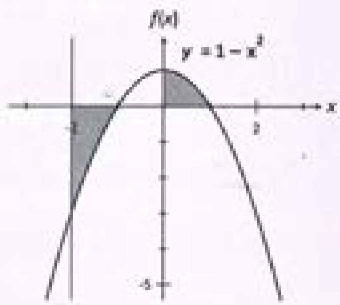

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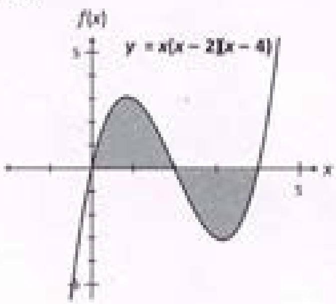

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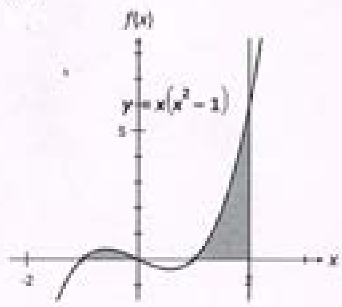

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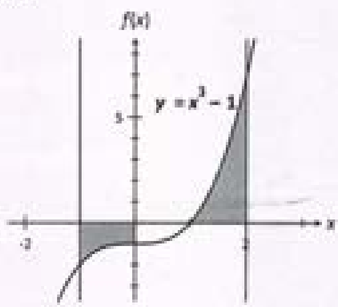


2. Find the area of the region trapped between each of the following curves and x-axis.
   (a) $y = x(x - 3)$ (b) $y = -x(x + 2)$
   (c) $y = (x - 1)(x - 2)(x + 3)$ (d) $y = -(x + 1)(x - 2)(x - 3)$

3. Find the area of the region trapped between given curve, the x-axis and the given lines:
   (a) $y = (x - 1)(x - 3)$, $x = 0$ and $x = 2$ (b) $y = (x + 1)(x + 3)$, $x = -4$ and $x = 0$
   (c) $y = x(x + 1)(x - 2)$, $x = -2$ and $x = 2$ (d) $y = -x(x + 1)(x + 2)$, $x = -3$ and $x = 2$.

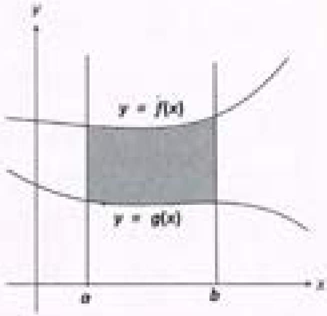
4. Find the area of the region trapped between given curve, the x-axis and the given lines:
   (a) $y = -2 + e^x$; $x = -1$, $x = 2$ (b) $y = \sin(x)$; $x = -\pi/2$ and $x = \pi$
   (c) $y = \tan(\pi x)$; $x = -1/4$ and $x = 1/4$ (d) $y = 1 - \sin(x/2)$; $x = -\pi$ and $x = \pi$

5. Find the area of the region trapped between given curve and the given lines:
   (a) $y = x^2$; $y = 0$ and $y = 2$ (b) $y = x^2 - 1$; $y = 0$ and $y = 1$
   (c) $y = (x - 1)^2$; left of the y-axis, $y = 1$ and $y = 4$
   (d) $y = (x - 1)^2$; $y = 0$ and $y = 2$

9.3 Area of Regions Involving Two Curves

- If, in the interval, $a \le x \le b, f(x) \ge g(x)$ for all values of x, then, the area of the region trapped between $y = f(x)$ and $y = g(x)$, and the lines $x = a$ and $x = b$ is given by:

$$A = \int_a^b f(x) - g(x)\ dx$$



- Note the pattern:

$$A = \int_a^b f(x) - g(x)\ dx$$

$f(x)$: curve that is on "top"

$g(x)$: curve that is at the bottom

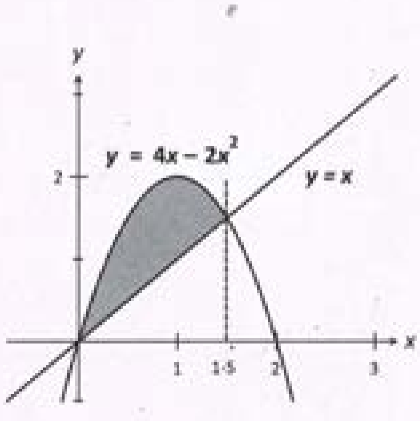
- *Caution:* This formula only works if $y = f(x)$ is always "on top" of $y = g(x)$ in the interval $a \le x \le b$.

- Where the use of a CAS calculator is permitted: the area of the region trapped between the curves $y = f(x)$ and $y = g(x)$, and the lines $x = a$ and $x = b$ is given by $A = \int_a^b |f(x) - g(x)|\ dx$.
  - In this case, it is not necessary to locate the relative positions of the two curves.

Example 9.6

Without the use of CAS calculator, find the area of the shaded region.

Solution:



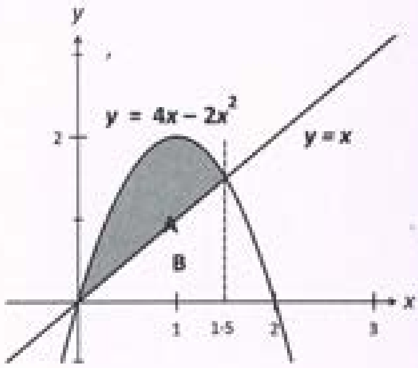
$$\text{Area} = \int_0^{\frac{3}{2}} (4x - 2x^2) - x\ dx = \int_0^{\frac{3}{2}} 3x - 2x^2\ dx$$

$$= \left[\frac{3x^2}{2} - \frac{2x^3}{3}\right]_0^{\frac{3}{2}}$$

$$= \left[\frac{3}{2} \times \frac{9}{4} - \frac{2}{3} \times \frac{27}{8}\right] = \frac{9}{8}.$$

Alternative Solution:

Let A be the region trapped between $y = 4x - 2x^2$, the lines $x = 0$ and $x = \frac{3}{2}$ and the x-axis and B be the region trapped between $y = x$ and the lines $x = 0$ and $x = \frac{3}{2}$ and the x-axis.



Then, Area of Shaded Region

$= \text{Area of A} - \text{Area of B}:$

$$= \int_0^{\frac{3}{2}} 4x - 2x^2\ dx - [\tfrac{1}{2} \times \tfrac{3}{2} \times \tfrac{3}{2}]$$

$$= \left[2x^2 - \frac{2x^3}{3}\right]_0^{\frac{3}{2}} - \frac{9}{8}$$

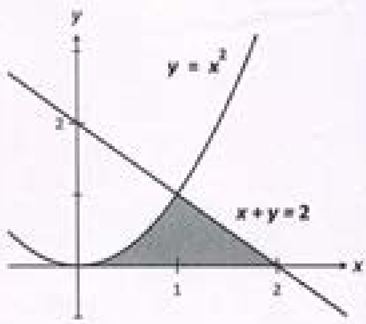
$$= \left[2 \times \frac{9}{4} - \frac{2}{3} \times \frac{27}{8}\right] - \frac{9}{8} = \frac{9}{8}.$$

Note:

- *In the second solution, the method of region subtraction is used.*

Example 9.7

Without the use of a CAS/graphic calculator, find the area of the shaded region.



Solution:

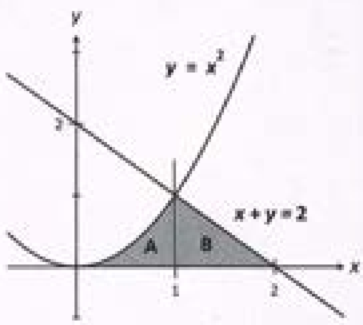
The shaded region can be divided into two parts, A and B, as shown in the accompanying diagram.

Area of shaded region = Area of A + Area of B

$$= \int_0^1 x^2\ dx + [\tfrac{1}{2} \times 1 \times 1]$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \frac{1}{2}$$

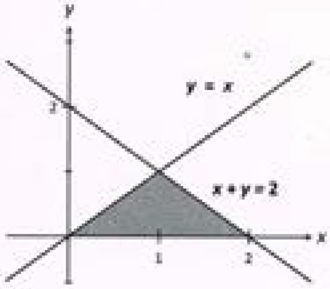
$$= \frac{1}{3} + \frac{1}{2} = \frac{2}{3}.$$



Exercise 9.3

1. Find the area of the shaded region.

(a)



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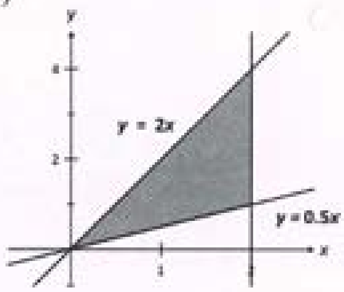

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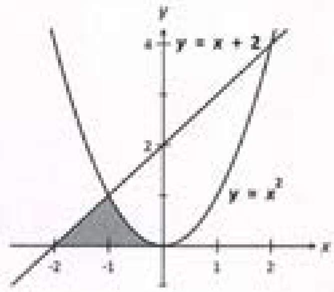

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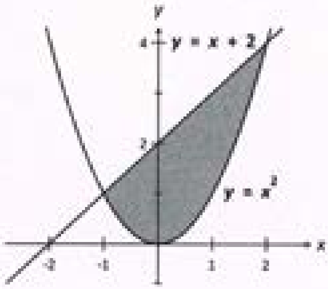

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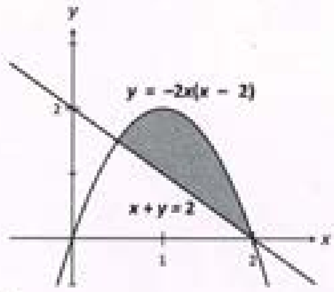

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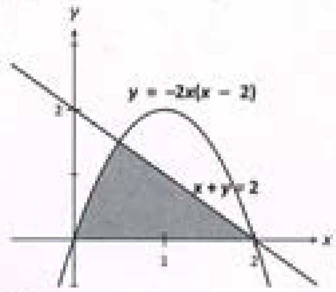

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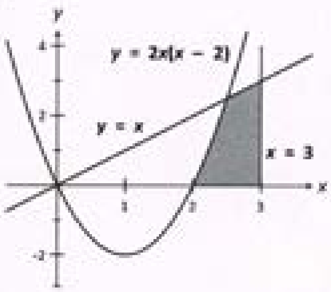

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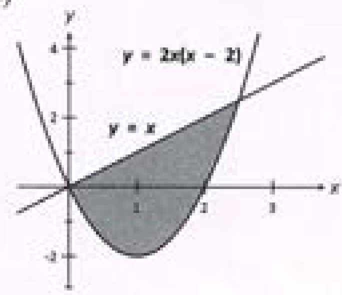

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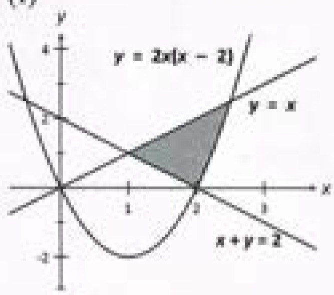
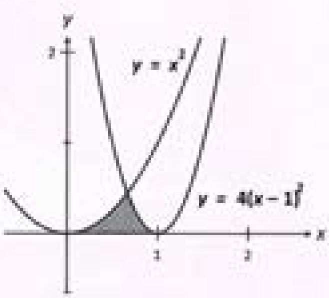
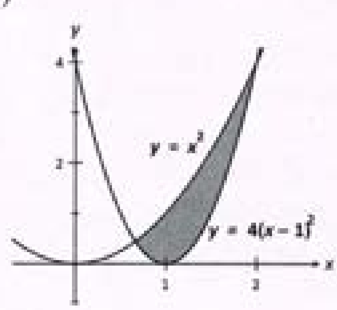


2. Use a calculus method involving the use of integrals to find the area of the shaded region.

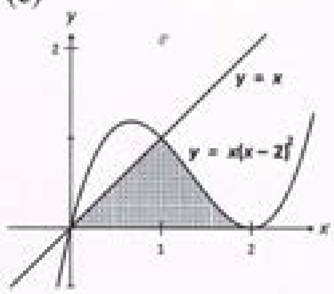
(a)



(b)

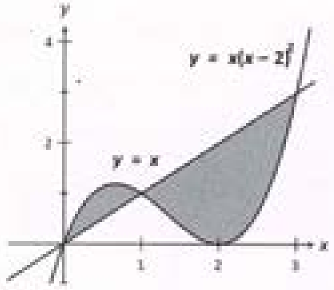


(c)

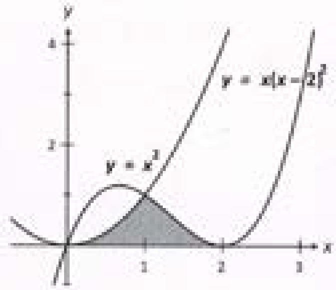


2.

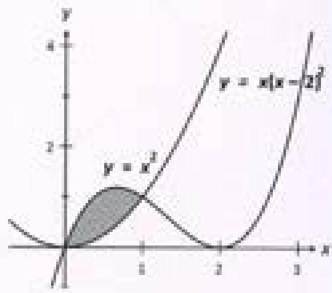
(d)



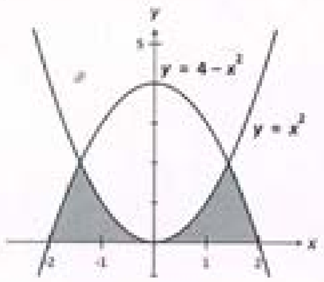
(e)



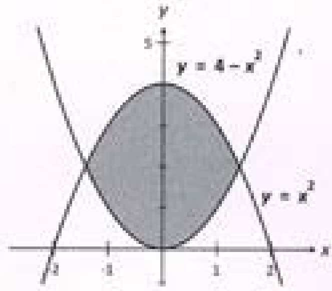
(f)



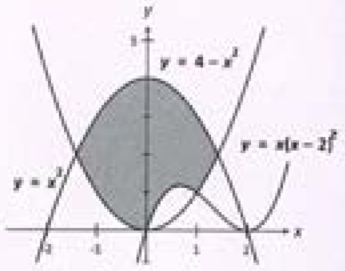
(g)



(h)



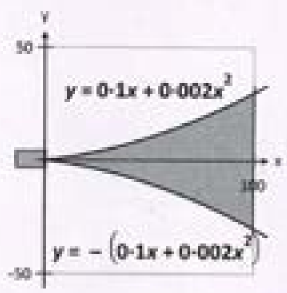
(i)



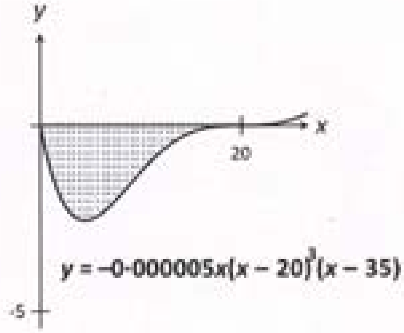
3. Use a calculus method to find the area of the region trapped by the curves:
   (a) $y = x^2$ and $y = -x + 2$
   (b) $y = -2x(x + 2)$ and $y = x + 2$
   (c) $y = 2x(x + 2)$ and $y = x + 2$
   (d) $y = x(x + 2)^2$ and $y = x$
   (e) $y = 2x^2$ and $y = x(x + 2)$
   (f) $y = -x^2$ and $y = x^2 - 4$
   (g) $y = x(x - 2)^2$ and $y = x^2(x - 2)$
   (h) $y = x(x + 1)^2$ and $y = x^2(x + 1)$
   (i) $y = x - 1$, $x = 3 - y^2$
   (j) $y^2 = x^3$, $y^2 = 2 - x$

4. Use calculus to find the area of the region trapped between the following curves:
   (a) $y = 1 + \sin(x)$, the x-axis, the y-axis and the line $x = \frac{3\pi}{2}$
   (b) $y = \sin(x)$ and $y = \cos(x)$ for $\frac{\pi}{4} \le x \le \frac{3\pi}{4}$

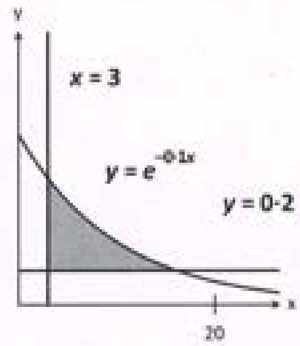
5. The wake left behind by a boat (up to a distance of 100 m behind the boat) as it travels through a lake, is indicated as the shaded region in the accompanying diagram. All measurements are in metres.
   (a) Find the width of the wake, 100 m behind the boat.
   (b) Find the area covered by the wake, from the point immediately behind the boat to a point 100 m behind the boat.



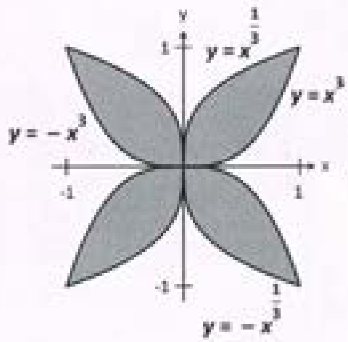
6. The cross-sectional profile of a gully is indicated as the shaded region in the accompanying diagram.
   (a) Find the cross-sectional area of this gully.
   (b) If the gully has a uniform cross-section along a length of 30 m, find the capacity of the gully. [1 m^3 = 1 kL]



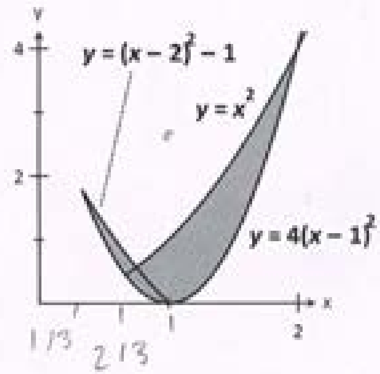
7. The accompanying diagram shows the cross-sectional profile of a block of land. The shaded region needs to be excavated. The block has a uniform cross-section along a length of 20 m. All measurements are in metres. How much earth, in m^3, needs to be excavated?



8. The accompanying diagram shows the logo of a particular company. The logo consists of four petals as shown. All measurements are in metres. Find the total area of the petals.



9. The accompanying diagram (shaded portion) shows the logo used to promote a certain product. Find the area of the logo. All measurements are in centimetres.



9.4 Area Functions

- Let $y = f(x)$ be continuous in the interval $a \le x \le b$, then

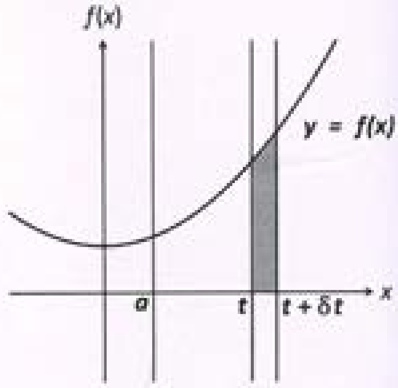
  $F(t) = \int_a^t f(x)\,dx$ represents the sum of the signed areas of the region trapped between the curve $y = f(x)$, the lines $x = a$, $x = t$ and the x-axis.

- By the Fundamental Theorem of Calculus, $F'(t) = \dfrac{d}{dt}\left[\int_a^t f(x)\,dx\right] = f(t)$.

  That is, the derivative of the area function $\int_a^t f(x)\,dx$ at $x = t$ is the value of its curve function $f(x)$ evaluated at $x = t$.

9.4.1 The Fundamental Theorem of Calculus revisited

- Consider the curve $y = f(x)$ which is continuous everywhere.
- Define the area function $A(t) = \int_a^t f(x)\,dx$.
- For $x = t + \delta t$: $\quad A(t + \delta t) = \int_a^{t+\delta t} f(x)\,dx$.



- Consider the strip trapped between the curve, the x-axis and the lines $x = t$ and $x = t + \delta t$.
  - The area of this strip is given by: $A(t + \delta t) - A(t)$.
  - The area of this strip may be approximated by the area of the inscribed rectangle and the circumscribed rectangle.
  - Hence: $f(t) \times \delta t < A(t + \delta t) - A(t) < f(t + \delta t) \times \delta t$

    $$f(t) < \frac{A(t+\delta t) - A(t)}{\delta t} < f(t + \delta t)$$

  - As $\delta t \to 0$, $\dfrac{A(t+\delta t) - A(t)}{\delta t} \to f(t)$.

    That is: $\displaystyle \lim_{\delta t \to 0}\left[\frac{A(t+\delta t) - A(t)}{\delta t}\right] = f(t)$

  - Hence: $\dfrac{dA}{dt} = f(t) \quad \Rightarrow \quad \dfrac{d}{dt}\left[\int_a^t f(x)\,dx\right] = f(t)$.

Example 9.8

Let $F(t) = \int_0^t x^3 - 7x^2 + 10x \, dx$ where for $0 \le t \le 8$. Calculate:

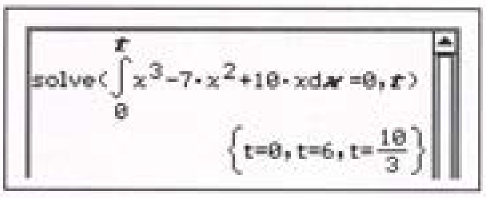
(a) $F(t)$ and hence $F(2)$ and $F(5)$.
(b) t if $F(t) = 0$.
(c) the global minimum and global maximum values for $F(t)$.

Solution:

(a) $F(t) = \dfrac{t^4}{4} - \dfrac{7t^3}{3} + 5t^2$

$F(2) = \dfrac{16}{3}$ and $F(5) = -\dfrac{125}{12}$

(b)
$$\frac{t^4}{4} - \frac{7t^3}{3} + 5t^2 = 0$$
$$3t^4 - 28t^3 + 60t^2 = 0$$
$$t^2(3t^2 - 28t + 60) = 0$$
$$t^2(3t - 10)(t - 6) = 0$$
$$t = 0, \frac{10}{3}, 6$$



(c) $F'(t) = t^3 - 7t^2 + 10t$

$F'(t) = 0 \Rightarrow t(t^2 - 7t + 10) = 0$

$t(t-2)(t-5) = 0$

$t = 0, 2, 5$

$F''(t) = 3t^2 - 14t + 10$

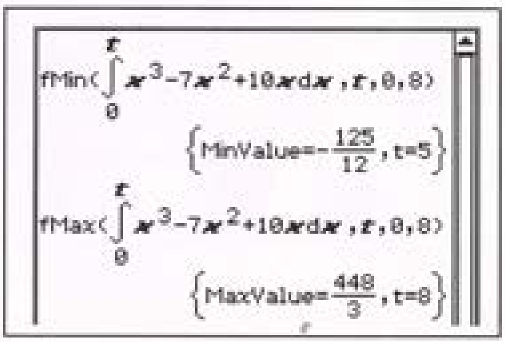
$F''(0) > 0$, $F''(2) < 0$ and $F''(5) > 0$

$\Rightarrow$ Local Minimum value for $F(t)$ is

$F(5) = -\dfrac{125}{12}$.

Local Maximum value for $F(t)$ is

$F(2) = \dfrac{16}{3}$.



Check endpoints: $F(0) = 0$ and $F(8) = \int_0^8 x^3 - 7x^2 + 10x \, dx = \dfrac{448}{3}$.

Hence, Global Minimum value is $-\dfrac{125}{12}$.

Global Maximum value is $\dfrac{448}{3}$.

Exercise 9.4

1. Let $F(t)=\int_0^t -x+2\ dx$ where for $0\le t\le 4$. Without the use of a calculator, find:
   (a) $F(t)$ and hence $F(0)$ and $F(2)$ (b) t if $F(t)=0$
   (c) the local maximum value for $F(t)$.

2. Let $F(t)=\int_{-4}^t 2x-4\ dx$ where for $-4\le t\le 8$. Without the use of a calculator, find:
   (a) $F(t)$ and hence $F(-2)$ and $F(8)$ (b) t if $F(t)=-20$
   (c) the global minimum and global maximum values for $F(t)$.

3. Let $F(t)=\int_{-2}^t f(x)\ dx$ where $f(x)=10-2x$, for $-2\le t\le 5$. Without the use of a calculator : (a) Find $F'(t)$ (b) Find $F'(2)$ and $f(2)$
   (c) Comment on your answers in (b).

4. Let $F(t)=\int_{-1}^t f(x)\ dx$ where $f(x)=x^2-4$, for $-1\le t\le 6$. Without the use of a calculator : (a) Find $F'(t)$ (b) Find $F'(4)$ and $f(4)$
   (c) Comment on your answers in (b).

5. The curve $y=f(x)$ is continuous everywhere. Let $F(t)=\int_0^t f(x)\ dx$.

   Prove that $F(t)$ has local extrema at the points where $y=f(x)$ has roots.

6. Let $F(t)=\int_{-5}^t -x^3+7x^2+8x\ dx$ where for $-1\le t\le 10$.

   Calculate the global minimum and maximum values for $F(t)$.

7. Let $F(t)=\int_{-6}^t -x^4+2x^3+5x^2-6x\ dx$ where for $-6\le t\le 6$.

   Calculate the global minimum and maximum values for $F(t)$.

9.5 Integration & Rate of Change

- Given $P = f(t)$, $\frac{dP}{dt}$ measures the instantaneous rate of change of P with respect to t.
- Hence, given $\frac{dP}{dt}$, P can be obtained by using $P = \int \frac{dP}{dt}\, dt$.

Example 9.9

The instantaneous rate of change of temperature (θ degrees Celsius) with respect to time (t hours) is given by $\frac{d\theta}{dt} = e^{-0.01t}$. Find the temperature of the body in terms of time t, given that the initial temperature of the body was 50 Celsius. Hence, find the temperature of the body after 5 hours.

Solution:

$\frac{d\theta}{dt} = e^{-0.01t}$ $\Rightarrow$ $\theta = \int e^{-0.01t}\, dt = \frac{e^{-0.01t}}{-0.01} + C$

Thus, $\theta = -100e^{-0.01t} + C$

When $t = 0$, $\theta = 50$, $\Rightarrow$ $50 = -100 + C$ $\Rightarrow$ $C = 150$.

Therefore $\theta = -100e^{-0.01t} + 150$.

When $t = 5$, $\theta = -100e^{-0.01(5)} + 150 = 54.9$ Celsius.

Example 9.10

The instantaneous rate with which effluent is discharged into a recycling pond is modelled by $\frac{dV}{dt} = \frac{1}{(1+t)^2}$, where V is the volume of effluent discharged in kilolitres and t is time in hours. Find the volume of effluent discharged in the first four hours.

Solution:

$\frac{dV}{dt} = \frac{1}{(1+t)^2}$ $\Rightarrow$ $V = \int (1+t)^{-2}\, dt = \frac{-1}{(1+t)} + C$

When $t = 0$, $V(0) = -1 + C$. When $t = 4$, $V(4) = -\frac{1}{5} + C$.

Hence, the volume of effluent discharged into the pond in the first four hours is given by $V(4) - V(0) = (-\frac{1}{5} + C) - (-1 + C) = \frac{4}{5}$ kL.

Note:

- *The volume of effluent discharged into the pond in the first four hours is $V(4) - V(0)$. This is the* ***net change*** *in the volume in the interval $0 \le t \le 4$. But, using the Fundamental Theorem of Calculus $\int_0^4 \frac{dV}{dt}\, dt = V(4) - V(0)$. Hence, the net change in V, in the interval $0 \le t \le 4$ is given by $\int_0^4 \frac{dV}{dt}\, dt$.*

9.5.1 Net Change

- If $\frac{dP}{dt}$ is the instantaneous rate of change of variable P with respect to t,

  then the ***net change*** in P, in the interval $a \le t \le b$, is given by $\int_a^b \frac{dP}{dt}\,dt$.

Example 9.11

The instantaneous rate with which, N, the number of carriers of a virus within a population changes with time t (days) is modelled by $\frac{dN}{dt} = \frac{-10}{\sqrt{1+t}}$. Find the net change in the number of carriers in the (a) first 8 days (b) 8th day.

Solution:

(a) Net change in N in the first 8 days = $\int_0^8 \frac{-10}{\sqrt{1+t}}\,dt = -10\int_0^8 (1+t)^{-1/2}\,dt = -40.$

(b) Net change in N in the 8th day = $\int_7^8 \frac{-10}{\sqrt{1+t}}\,dt = -10\int_7^8 (1+t)^{-1/2}\,dt = -3.43 \approx -3.$

Exercise 9.5

1. Given that $\frac{dA}{dt} = 4t + 2$, find A given that when $t = 0$, $A = 1$.

2. Given that $\frac{dQ}{dt} = (1 - t)^2$, find the net change in Q in the interval $0 \le t \le 4$.

3. Given that $\frac{dV}{dt} = \sqrt{1+2t}$, find the net change in V when t changes from $t = 2$ to $t = 4$.

4. Given that $\frac{dM}{dt} = \frac{1}{e^{2t}}$, find the net change in M in the interval $0 \le t \le 2$.

5. Given that $\frac{dQ}{dt} = \sin(t)$, find the net change in Q when t changes from 0 to π.

6. Find the net change in V in the interval $1 \le t \le 5$, given that $\frac{dV}{dt} = \frac{1}{\sqrt{t}}$.

7. Find the net change and hence the average rate of change in L in the interval $2 \le t \le 4$, given that $\frac{dL}{dt} = \frac{1}{(1+t)^2}$.

8. The instantaneous rate with which the temperature, θ degrees Celsius, of a body changes with respect to time, t minutes, is modelled by $\frac{d\theta}{dt} = 2 - 0.05t$.
   (a) Find when the maximum temperature of the body occurred.
   (b) Find the net change in temperature in the (i) first 5 minutes (ii) 5th minute.

9. The instantaneous rate with which the average price, P cents, of a share on the Stock Exchange, changes with respect to time, t months, is modelled by $\frac{dP}{dt} = t^2 - 3t - 10$, for $0 \le t \le 12$. The initial price of the share was 60 cents.
   (a) Find the lowest price of this share and when this occurred.
   (b) Find the net change in price in the (i) first 6 months (ii) 7th month.

10. The instantaneous rate with which the concentration C, mg/kL, of a chemical compound in a river system, changes with respect to time, t weeks, is modelled by $\frac{dC}{dt} = \frac{1}{(t+0.5)^2} - 2t$, for $t \ge 0$. The initial concentration was 9.3 mg/kL.
   (a) Find the net change in concentration in the first week.
   (b) Find the maximum concentration and when this occurred.

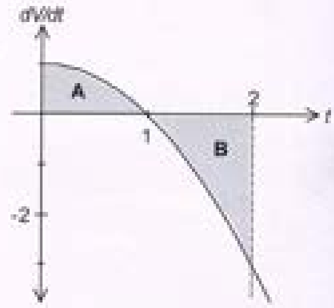
11. The instantaneous rate with which water flows into an ornamental pond, is modelled by $\frac{dV}{dt} = (t+1)(t+3)(4-t)$, for $t \ge 0$, where V is the volume of water in the pond in kL and t is time in hours.
   (a) Find when the water stops flowing into the pond.
   (b) Find the amount of water that has flowed into the pond in the first hour.
   (c) Find the volume of water that has flowed into the pond.

12. A rechargeable battery has a voltage of 9 volts when fully charged. When the battery is used to run an electronic toy, the voltage V volts, remains at 9 volts for 30 minutes and then the voltage decreases instantaneously, at a rate modelled by $\frac{dV}{dt} = -0.2e^{-0.02t}$.
   (a) Find the net change in the battery voltage after the toy has been in use for 40 minutes.
   *(b) Find how long the battery can be used to run this toy, if a minimum voltage of 8 volts is required.

13. The marginal cost for producing x items of a product is given by $\frac{dC}{dx} = 0.04x$, where $\$C$ is the cost of producing x items of a product.
   (a) Given that the fixed cost is \$20, find the cost of producing 100 of these items.
   (b) Find the net change in cost if the number of items produced is changed from 100 to 200.

14. The marginal profit associated with the sale of x items of a product is given by $\frac{dP}{dx} = -0.000\,27x^2 + 0.18x - 3.6$, where \$$P$ is the profit associated with the sale of x items of this product.
    (a) Given that there is a loss of \$50 if no items are sold, find the profit associated with the sale of 100 items.
    (b) Find the net change in profit if the number of items sold is changed from 100 to 200.

15. The instantaneous rate with which the amount of liquid, V litres, in a holding tank, changes with respect to time t minutes, is modelled by $\frac{dV}{dt} = 1 - t^2$. The sketch of $\frac{dV}{dt}$ against t is shown in the accompanying diagram.



    (a) Explain what happens at $t = 1$ minute.
    (b) Find the area of region A and interpret your answer.
    (c) Find the area of region B and interpret your answer.
    (d) Find the amount of liquid in the tank after 2 minutes, if initially there were 5 litres in the tank.

16. Given that $\frac{dT}{dx} = 15\sin\left[\frac{2\pi}{365}(x-100)\right] + 25$, find the net change in T (measured in degrees Celsius) corresponding to a change in x from $x = 90$ days to $x = 140$ days. Hence, find the average rate of change within the corresponding period of time.

17. The rate of change of pressure P (kiloPascal) at a given time t (hours) in a leaking tyre is modelled by $\frac{dP}{dt} = \frac{-200}{(1+t)^2}$. Find the rate of change of pressure when $t = 9$ hours. Find the net change and hence the average rate of change in tyre pressure in the interval $0 \le t \le 9$ hours. Assume the leak has not been fixed in this time interval!

18. The rate of change of volume V (cm^3) at a given time t (seconds) of an inflatable toy is modelled by $\frac{dV}{dt} = \frac{100\,000}{(1+t)^2}$. Find the rate of change of volume when $t = 1$ minute.

    Find the net change and hence the average rate of change in the volume of the toy in the interval $0 \le t \le 2$ minutes.

19. The rate at which the speed of contraction of a muscle V changes in relation to the load acting on the muscle F is modelled by $\frac{dV}{dF} = \frac{-b(a+F_0)}{(F+a)^2}$, where a, b and F_0 are constants.

    Find the net change in V corresponding to a change in F from F_0 to $4F_0$.

10 Rectilinear Motion

10.1 Rectilinear Motion and Differentiation

- In this section, attention is focused on objects that move along straight lines. Such objects are said to experience rectilinear motion.

- Consider an object P moving along a straight line.
  Let its displacement from a fixed point O at time t be s.

- Then, $\dfrac{ds}{dt}$ represents the instantaneous rate of change of displacement with respect to time t. This is termed the ***instantaneous velocity*** of P, usually denoted v.

- Then, $\dfrac{dv}{dt} \equiv \dfrac{d^2s}{dt^2}$ represents the instantaneous rate of change of velocity with respect to time t. This is termed the ***instantaneous acceleration*** of P, usually denoted a.

- The relationship between displacement s, velocity v and acceleration a, can be represented schematically as follows.
  displacement $\xrightarrow{\text{differentiate}}$ velocity $\xrightarrow{\text{differentiate}}$ acceleration

- The speed of the body at time $t = |v|$.

- The average velocity of a body over a given interval of time is given by:
$$\text{average velocity} = \frac{\text{change in displacement}}{\text{time interval}}$$

- The average speed of a body over a given time interval is given by:
$$\text{average speed} = \frac{\text{distance travelled}}{\text{time interval}}$$

Hands On Task 10.1

In this task, we will match the physical state of a particle experiencing rectilinear motion with a mathematical condition.

Consider a particle P, moving in a straight line. Let its displacement from a fixed point O at time t be s. Let its velocity and acceleration at time t, be v and a respectively. Complete the following tables.

Physical State	**Mathematical condition relating to**		
	displacement s	**velocity v**	**acceleration a**
P is at O	$s = 0$	–	–
P returns to O			
P is at rest			
P is instantaneously at rest			
P starts off from rest from O			
P comes to rest at O			
P achieves maximum displacement			
P achieves maximum speed			
P reverses direction			

Mathematical statement	**Physical state**
$s = 0$	
$s > 0$	
$s < 0$	
$v = 0$	
$v < 0$	
$v > 0$	
$s > 0$ and $v > 0$	
$s > 0$ and $v < 0$	
$s < 0$ and $v > 0$	
$s < 0$ and $v < 0$	
$a = 0$	

Example 10.1

A particle P moves along a straight line. Its displacement, s metres, t seconds after passing a fixed point O, is given by, $s = t(t-2)(t-5) \equiv t^3 - 7t^2 + 10t$, for $t \geq 0$.

(a) Find when the particle returns to O.
(b) Find the velocity of P when it returns to O.
(c) Find when the body is instantaneously at rest.

Solution:

(a) Particle returns to O when $s = 0$.
Hence, $t(t-2)(t-5) = 0 \Rightarrow t = 0, 2, 5$
That is, P returns to O when $t = 2$ seconds and when $t = 5$ seconds.

(b) Velocity $v = \dfrac{ds}{dt} = 3t^2 - 14t + 10$
When $t = 2$, $v = -6$. When $t = 5$, $v = 15$
Hence, velocity of P when it returns to O is $-6\ \text{ms}^{-1}$ and $15\ \text{ms}^{-1}$.

(c) Body is at rest when $v = 0 \Rightarrow 3t^2 - 14t + 10 = 0 \Rightarrow t = 0.88, 3.79$
That is, P is instantaneously at rest when $t = 0.88$ s and 3.79 s.

Example 10.2

A particle P, moves along a straight line. Its displacement, s metres, from a fixed point O at time t seconds, is given by, $s = -t^3 + 9t^2 - 23t + 15 \equiv (1-t)(t-3)(t-5)$, for $0 \leq t \leq 5$.

(a) Find the initial displacement and initial velocity of P.
(b) Find the acceleration of P when $t = 1$ second.
(c) Find when the body achieves maximum speed and give the maximum speed.

Solution:

(a) When $t = 0$, $s = 15$
Velocity, $v = \dfrac{ds}{dt} = -3t^2 + 18t - 23$
When $t = 0$, $v = -23$
Hence, initial displacement is 15 m and initial velocity is $-23\ \text{ms}^{-1}$.

(b) Acceleration, $a = \dfrac{dv}{dt} = -6t + 18$
When $t = 1$, $a = 12$
Hence, acceleration when $t = 1$ is $12\ \text{ms}^{-2}$.

(c) v has a local maximum when $a = 0$. $\Rightarrow t = -18/(-6) = 3$ s.
Hence local maximum for $v = 4\ \text{ms}^{-1}$. $\Rightarrow$ local maximum speed is $4\ \text{ms}^{-1}$.
When $t = 0$, $v = -23\ \text{ms}^{-1}$, hence the speed is $23\ \text{ms}^{-1}$.
When $t = 5$, $v = -8\ \text{ms}^{-1}$, hence the speed is $8\ \text{ms}^{-1}$.
Hence, the maximum speed is $23\ \text{ms}^{-1}$ when $t = 0$.

Exercise 10. 1

1. A particle P moves along a straight line. Its displacement, s metres, from a fixed point O, at time t seconds, is given by, $s = (t - 1)(t - 4)$ for $0 \le t \le 7$.
   (a) Find the initial displacement of P.
   (b) Find when the particle is at O and its velocity when it is at O.
   (c) Find when the body is instantaneously at rest.
   (d) How long was P to the left of O?

2. A particle P moves along a straight line. Its displacement, s metres, from a fixed point O, at time t seconds, is given by, $s = (t - 1)(3 - t)(t - 4) \equiv -t^3 + 8t^2 - 19t + 12$ for $0 \le t \le 10$.
   (a) Find when the particle is at O.
   (b) Find when P reverses direction.
   (c) Find how long was P to the right of O?
   (d) Find the acceleration of P when P is instantaneously at rest.
   *(e) Find the distance travelled in the first 3 seconds.

3. A particle P moves along a straight line. Its velocity, v ms^{-1}, at time t seconds, is given by, $v = 2t - 8$ for $0 \le t \le 6$.
   (a) Find the initial velocity of P.
   (b) Find the average acceleration of P in the first 6 seconds.
   (c) Find the instantaneous acceleration of P. Comment on your answer.

4. A particle P moves along a straight line. Its velocity, v ms^{-1}, at time t seconds, is given by $v = t^2 - 8t + 12$ for $0 \le t \le 10$. Use an analytical method to:
   (a) find when P reverses direction.
   (b) find the velocity of P when its acceleration is zero.
   (c) find the acceleration of P when it is instantaneously at rest.
   *(d) find the maximum speed of P and when it occurs.

5. A particle P moves along a straight line. Its displacement, s metres, from a fixed point O, at time t seconds, is given by, $s = (t - 1)^2(t - 4) \equiv t^3 - 6t^2 + 9t - 4$ for $0 \le t \le 10$.
   (a) Find when P returns to O.
   (b) Find the average velocity in the first 2 seconds.
   (c) Find the velocity when the acceleration is 0.
   *(d) Find the distance travelled in the first 2 seconds.
   *(e) Find the average speed in the first 2 seconds.

6. A particle P moves along a straight line. Its velocity, v ms^{-1}, at time t seconds, is given by $v = 4\,e^{-0.5t}$ for $t \ge 0$.
   (a) Find the initial velocity and acceleration for P.
   (b) Find the average acceleration for P in the interval $0 \le t \le 5$.
   (c) Find the acceleration of P when the velocity is 2 ms^{-1}.
   (d) Find the velocity of P when the acceleration is -0.5 ms^{-2}.
   (e) Describe the motion of P for large values of t.

7. A particle P moves along a straight line. Its velocity, v ms^{-1}, at time t seconds, is given by, $v = 10\, t\, e^{-t}$ for $t \geq 0$.
   *(a) Find the maximum speed attained by P.
   (b) Find the acceleration at time $t = 4$ seconds.

8. Particle P travels in a straight line such that its displacement x (metres) from a fixed point O, t seconds after passing O, is given by $x = e^{t} \sin(t)$. For $0 \leq t \leq \pi$, calculate:
   (a) when P is instantaneously at rest and its acceleration at this instant
   (b) the maximum velocity for P (in the positive direction).

9. A particle moves along the x-axis, and after t seconds, its position from the origin O is given by $x = 5 + 3\cos(2t) + 4\sin(2t)$.
   (a) Calculate when the body is momentarily at rest
   (b) prove that the acceleration of the particle is given by $a = 20 - 4x$.

10. A particle P moves along a straight line. Its displacement from a fixed point O, s metres, at time t seconds, is given by, $s = \dfrac{25t}{5t + 2}$ for $t \geq 0$.
   (a) Find the initial velocity of P.
   (b) Find the average velocity of P for $0 \leq t \leq 5$.
   *(c) Find the distance travelled in the first 5 seconds.
   (d) Find the average speed in the first 5 seconds.
   (e) Describe the motion of P for large values of t.

11. An object P is thrown vertically upwards and moves in a straight line. Its velocity, v ms^{-1}, at time t seconds, is given by, $v = 19.6 - 9.8t$ for $t \geq 0$.
   (a) Find the initial velocity of P.
   (b) Find when P is instantaneously at rest.
   (c) Find when P hits the ground and the velocity with which it hits the ground.
   (d) Find the acceleration of P and comment on your answer.

12. A ball Q is thrown down from a tall building and falls vertically downwards. Its velocity, v ms^{-1}, at time t seconds, is given by, $v = -9.8 - 9.8t$ for $t \geq 0$.
   (a) Find the initial velocity of Q.
   (b) Find the velocity of Q after 2 seconds.
   (c) Find the average acceleration of Q in the first 2 seconds.
   (d) Find the acceleration of Q and comment on your answer.

13. An object P is thrown vertically upwards. Its height, h metres, at time t seconds, is given by, $h = 10t - 4.9t^2$ for $t \geq 0$.
   (a) Find the initial velocity of P.
   (b) Find the highest point reached by P.
   (c) Find the velocity of P as it hits the ground.
   (d) Find the acceleration of P.

14. A projectile P is launched vertically upwards. Its height, h metres, at time t seconds, is given by, $h = 30t - 4.9t^2$ for $t \geq 0$.
    (a) Find the height after 5 seconds.
    (b) Find the velocity of P as it hits the ground.
    (c) Find when its velocity is 20 ms^{-1}.
    *(d) Find when its speed is 20 ms^{-1}.

15. A ball is projected vertically upwards from a height of 2 m. Its height, h metres, from the point of projection, at time t seconds, is given by, $h = 5t - 4.9t^2$ for $t \geq 0$.
    (a) Find the time it takes for the ball to hit the ground.
    (b) Find the total distance travelled by the ball before hitting the ground.
    (c) Find the velocity of the ball when it is 3 m above the ground.
    (d) Find the speed with which the ball hits the ground.

16. A bowling ball (lawn bowl) is rolled on the ground and moves in a straight line, Its distance, d metres, from a fixed point, after t seconds, before it comes to rest, is given by, $d = 5t - t^2$.
    (a) Find the initial velocity of the ball.
    (b) Find the distance it rolls before it comes to a stop.
    (c) Find the acceleration of the ball. Comment on your answer.

17. Damien lifts his foot off the accelerator pedal, and allows his car to coast to a stop. Assume that the car moves in a straight line. The distance travelled by the car, s metres, t seconds after Damien lifts his foot off the accelerator pedal, and before the car comes to a stop, is given by, $s = 12t - t^2$.
    (a) Find the time taken by the car to come to a complete stop.
    (b) A stationary car is 30 m directly ahead of Damien's car when the he lifts his foot off the accelerator pedal. Will Damien's car hit the stationary car?
    (c) Find the acceleration of the car and comment on your answer.

18. A car is travelling in a straight line. Its distance, s metres, from a fixed point O, at time t seconds, is given by, $s = 10 + 14t + \frac{1}{3}t^3$ for $t \geq 0$.
    (a) Find the initial velocity of the car.
    (b) Find when the velocity of the car is 20 ms^{-1}.
    (c) Find the acceleration of the car. Comment on your answer.
    (d) Find the average speed of the car in the first 4 seconds.

10.2 Rectilinear Motion and Integration

- Consider a body P moving along a straight line.
  Let its displacement from a fixed point at time t seconds be s metres.
- The instantaneous velocity $v = \dfrac{ds}{dt}$. Hence, $s = \int v \, dt$.
- The instantaneous acceleration $a = \dfrac{dv}{dt}$. Hence, $v = \int a \, dt$.
- The net change in displacement in the interval $a \le t \le b$ is $\int_a^b v \, dt$.
- The total distance travelled in the interval $a \le t \le b$ is the area of the region trapped between the curve $v = f(t)$, the lines $t = a$, $t = b$ and the t-axis.
  Where use of a CAS/graphic calculator is permitted,
  Total distance travelled = $\int_a^b |v| \, dt$.
- The relationship between displacement s, velocity v and acceleration a, can be represented schematically as shown below.

Example 10.3

A particle P moves along a straight line. Its acceleration, a ms^{-2}, t seconds after passing a fixed point O, is given by $a = 2t$. P starts off from O with a velocity of 1 ms^{-1}. Without the use of a CAS/graphic calculator:
(a) find its velocity after 2 seconds. (b) find its displacement after 2 seconds

Solution:

(a) $a = 2t \quad \Rightarrow \quad v = \int 2t \, dt = t^2 + C$

When $t = 0$, $v = 1 \quad \Rightarrow \quad 1 = C$

Hence, $v = t^2 + 1$

Therefore, when $t = 2$, $v = 5$ ms^{-1}

(b) $v = t^2 + 1 \quad \Rightarrow \quad s = \int t^2 + 1 \, dt = \dfrac{t^3}{3} + t + K$

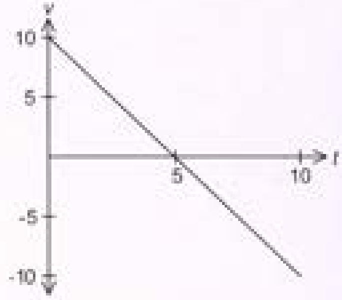
When $t = 0$, $s = 0 \quad \Rightarrow \quad 0 = K$

Hence, $s = \dfrac{t^3}{3} + t$

Therefore, when $t = 2$, $s = \dfrac{14}{3}$ m.

Example 10.4

A particle P moves along a straight line. The velocity-time graph of P (for $0 \le t \le 10$) is shown in the accompanying diagram. Velocity is measured in ms^{-1} and time in seconds. Without the use of a CAS/graphic calculator:



(a) find the acceleration of P when $t = 5$ seconds.
(b) find the change in displacement in the first 10 seconds.
(c) find the total distance travelled in the first 10 seconds.
(d) find the average speed in the first 10 seconds.

Solution:

(a) Acceleration of P when $t = 5$ seconds = gradient of v-t graph at $t = 5$.
$= 10/(-5) = -2 \text{ ms}^{-2}$

(b) Change in displacement $= \int_0^{10} v\, dt = \frac{1}{2} \times 5 \times 10 - \frac{1}{2} \times 5 \times 10 = 0$ m

(c) Total distance travelled $= \left[\int_0^5 v\, dt\right] + \left[-\int_5^{10} v\, dt\right] = 25 + 25 = 50$ m

(d) Average speed in the interval $0 \le t \le 10 = 50/10 = 5 \text{ ms}^{-1}$

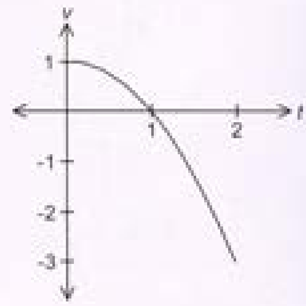
Example 10.5

A particle P travels in a straight line. Its velocity, $v \text{ ms}^{-1}$, t seconds after passing a fixed point O, is given by $v = -t^2 + 1$, for $0 \le t \le 2$.
(a) find the change in displacement within the first 2 seconds.
(b) find the total distance travelled in the first 2 seconds.

Solution:

(a) Change in displacement $= \int_0^2 -t^2 + 1\, dt = -\frac{2}{3}$ m.

(b) The sketch of $v = -t^2 + 1$, for $0 \le t \le 2$, is given in the accompanying diagram.

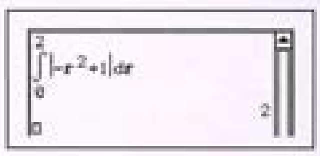


Total distance travelled $= \left[\int_0^1 -t^2 + 1\, dt\right] + \left[-\int_1^2 -t^2 + 1\, dt\right]$

$= \frac{2}{3} + \frac{4}{3} = 2$ m

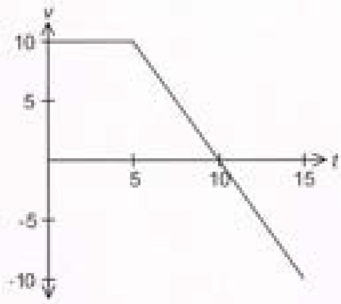
Alternatively:

Total distance travelled $= \left[\int_0^2 \left|-t^2 + 1\right|\, dt\right] = 2$ m.



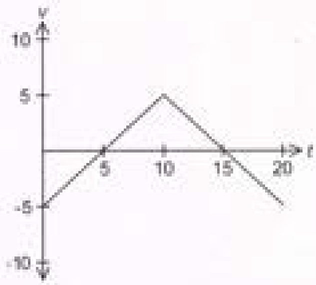
Exercise 10.2

1. A particle P moves along a straight line. The velocity-time graph of P is shown in the accompanying diagram. Velocity is measured in ms^{-1} and time in seconds. Without the use of a CAS/graphic calculator:



   (a) find the velocity of P when $t = 2$ seconds.
   (b) find the acceleration of P when $t = 12$ seconds.
   (c) find the change in displacement in the first 15 seconds.
   (d) find the total distance travelled in the first 15 seconds.
   (e) find the average speed in the first 15 seconds.

2. A particle P moves along a straight line. The velocity-time graph of P is shown in the accompanying diagram. Velocity is measured in ms^{-1} and time in seconds. Without the use of a CAS/graphic calculator:



   (a) find the velocity of P when $t = 5$ seconds.
   (b) find when P reverses direction
   (c) find the acceleration of P when $t = 15$ seconds.
   (d) find the total distance travelled in the first 20 seconds.
   (e) find the position of P in relation to the starting point at $t = 20$ seconds.

3. A particle P travels in a straight line. Its velocity, v ms^{-1}, t seconds after passing a fixed point O, is given by $v = t^2 + 2t + 3$. Without the use of a CAS/graphic calculator:
   (a) find the displacement of P after 2 seconds.
   (b) find the instantaneous acceleration of P at $t = 2$ seconds.

4. A particle P travels in a straight line. Its velocity, v ms^{-1}, at time t seconds, is given by $v = -t^2 + 5t - 4$. P is initially 1 m from a fixed point O. Without the use of a CAS/graphic calculator:
   (a) find the displacement of P from O after 5 seconds.
   (b) find the initial instantaneous acceleration of P.

5. The velocity of a particle P experiencing rectilinear motion is given by $v = \pi \sin(\pi t)$ ms^{-1}. Calculate:
   (a) the velocity of P when $t = 2$ seconds
   (b) the net change in displacement in P in the first two seconds
   (c) the distance travelled by P in the first two seconds
   (d) the average speed of P during the first two seconds.

6. The acceleration of a particle P undergoing rectilinear motion is given by $a = -4\pi^2 \sin(2\pi t)$ ms^{-2}. The initial velocity of P is 2π ms^{-1}. Calculate:
   (a) the velocity of P when $t = 1.5$ seconds
   (b) the net change in displacement in P in the first one and a half seconds
   (c) the distance travelled by P in the first one and a half seconds
   (d) the average speed of P during the first one and a half seconds
   (e) the average acceleration of P during the first one and a half seconds.

7. A particle P starts off from a fixed point O, with an initial velocity of 1 ms^{-1}.
Its acceleration, a ms^{-2}, after t seconds is given by $a = e^{-t}$. Without the use of a CAS/graphic calculator find:
(a) the velocity of P after 4 seconds. (b) the displacement of P after 4 seconds.
(c) the change in displacement in the first 4 seconds.
(d) the total distance travelled in the first 4 seconds.

8. A particle P travels in a straight line. Its initial displacement from a fixed point O, and initial velocity, are -1 m and 6 ms^{-1} respectively. Its acceleration, a ms^{-2}, t seconds after passing O is given by $a = \dfrac{-10}{(t+1)^2}$.
(a) Find the velocity of P after 3 seconds.
(b) Discuss the motion of P for large values of t.
(c) Find the change in displacement in the 4th second.
*(d) Find the total distance travelled in the first 4 seconds.
(e) Find the magnitude of the average acceleration in the first 3 seconds.

9. A particle P travels in a straight line. Its velocity as it passes a fixed point O, is 2 ms^{-1}.
Its acceleration, $\dfrac{d^2x}{dt^2}$ ms^{-2}, t seconds after passing O is given by $\dfrac{d^2x}{dt^2} = 6t - 6$. Find the:
(a) velocity of P after 2 seconds *(b) maximum displacement of P for $0 \le t \le 2$.
(c) change in displacement in the first 2 seconds.
*(d) total distance travelled in the first 2 seconds.
(e) magnitude of the average acceleration in the first 2 seconds.

10. A particle P travels in a straight line. Its velocity as it passes a fixed point O, is -3 ms^{-1}.
Its acceleration, $\dfrac{d^2x}{dt^2}$ ms^{-2}, t seconds after passing O is given by $\dfrac{d^2x}{dt^2} = -6t + 8$. Find:
(a) when P changes direction. *(b) the maximum displacement of P for $0 \le t \le 3$.
(c) the change in displacement in the first 3 seconds.
(d) the total distance travelled in the first 3 seconds.
(e) the average speed in the first 3 seconds.

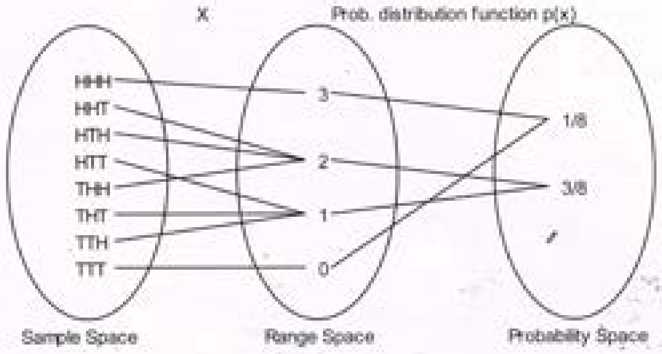
11. A particle P moves along a straight line. Its velocity, $\dfrac{dx}{dt}$ ms^{-1}, t seconds after passing a fixed point O, is given by $\dfrac{dx}{dt} = kt$, where k is a constant. Find the value of k, given that the net change in displacement in the first 2 seconds is 6 m.

12. A particle P moves along a straight line. Its velocity, $\dfrac{dx}{dt}$ ms^{-1}, t seconds after passing a fixed point O, is given by $\dfrac{dx}{dt} = kt + C$, where k and C are constants. The change in displacement in the first 4 seconds is 8 m. P reverses direction at $t = 3$ seconds.
Find the values of k and C.

11 Discrete Random Variables

11.1 Definitions

- A *Discrete Random Variable* (DRV) is a function that assigns a number (not necessarily a whole number) to each outcome of a statistical experiment.
- Consider an experiment where 3 identical fair coins are tossed. The sample space is S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}.
- Define DRV X: Number of heads in a toss of the 3 coins. The diagram below shows schematically the numbers that are assigned by X to each member of the sample space.



- Clearly the numbers assumed by X are 0, 1, 2 and 3. We write, X = 0, 1, 2, 3. The set formed by the values taken by X is called the range space.
- The probability of X assuming a value in its range space is equal to the probability of its equivalent event. For example, P(X = 3) = P(HHH) = 1/8.
- The *probability distribution* for a DRV X details the probability of X assuming each value within its range. The table below describes $P(X = x)$, the probability distribution for X. Note that the sum of all the probabilities is exactly one.

$X = x$	Equivalent event	$P(X = x) = p(x)$	$P(X \le x)$
3	HHH	$\frac{1}{8}$	$\frac{1}{8}$
2	HTH or THH or HHT	$\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$	$\frac{1}{8}+\frac{3}{8}=\frac{1}{2}$
1	HTT or TTH or THT	$\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$	$\frac{1}{2}+\frac{3}{8}=\frac{7}{8}$
0	TTT	$\frac{1}{8}$	$\frac{7}{8}+\frac{1}{8}=1$
	Total	1	

- The *probability distribution (mass) function* for X, $p(x)$, is a function that assigns each number that X takes, the probability of its equivalent event.
- The fourth column, $P(X \le x)$, describes the cumulative distribution for X.

11.2 Properties of the Probability Distribution Function

- Let $P(X = x) = p(x)$.
  $p(x)$ is a *probability distribution function* or *probability mass function* for a discrete random variable X if:
  - $p(x) \geq 0$ for all x in the range space of X
  - $\sum p(x) = 1$.

  In words, the individual probabilities are non-negative and add up to one.
  Note that: $\sum p(x)$ is read as the sum of all values of $p(x)$.

Example 11.1

Determine with reasons if each of the $p(x)$ as described are probability distribution functions.

(a)

x	0	1	3	5
$p(x)$	−0.1	0.1	0.4	0.6

(b)

x	−3	2	1	4
$p(x)$	0.1	0.2	0.3	0.4

Solution:

(a) $p(x)$ is not a probability distribution function as $p(0) = -0.1$ is negative.

(b) $p(x)$ is a probability distribution function as the individual probabilities are non-negative and add up to one.

Example 11.2

Given that $P(X = x) = \dfrac{x}{10}$ where $x = 1, 2, 3, 4$, find: (a) $P(2 \leq X \leq 3)$ (b) $P(X \geq 2 \mid X \leq 3)$.

Solution:

(a) $P(2 \leq X \leq 3) = P(X = 2) + P(X = 3)$

$$= \frac{2}{10} + \frac{3}{10} = \frac{1}{2}$$

(b) Using the conditional rule for probability:

$$P(X \geq 2 \mid X \leq 3) = \frac{P(X \geq 2 \cap X \leq 3)}{P(X \leq 3)} = \frac{P(2 \leq X \leq 3)}{P(X \leq 3)}$$

$$= \frac{(\frac{1}{2})}{(\frac{6}{10})} = \frac{5}{6}$$

Example 11.3

A box contains 3 white and 2 red balls. 3 balls are drawn without replacement.
Define X: the number of red balls drawn. Determine:
(a) the probability distribution for X (b) the cumulative probability distribution for X.

Solution:

(a) Clearly X = 0, 1, 2.
The table below tabulates the probability distribution for X.

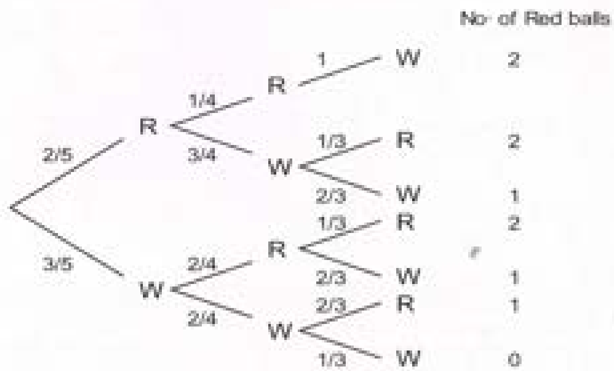
$X = x$	Equivalent event	$P(X = x)$
0	0 Red & 3 White Balls	$\dfrac{\binom{2}{0}\binom{3}{3}}{\binom{5}{3}} = \dfrac{1}{10}$
1	1 Red & 2 White Balls	$\dfrac{\binom{2}{1}\binom{3}{2}}{\binom{5}{3}} = \dfrac{6}{10}$
2	2 Red & 1 White Ball	$\dfrac{\binom{2}{2}\binom{3}{1}}{\binom{5}{3}} = \dfrac{3}{10}$

(b) Hence, the cumulative probability distribution for X is:

$$P(X \le 0) = \frac{1}{10},\ P(X \le 1) = \frac{7}{10},\ P(X \le 2) = 1$$

Alternative solution:

The tree diagram describing the situation is given in the accompanying diagram:
Let R: ball drawn is red,
W: ball drawn is white



The table below tabulates the probability distribution for X.

$X = x$	Equivalent event	$P(X = x)$
0	WWW	$\left(\frac{3}{5} \times \frac{2}{4} \times \frac{1}{3}\right) = \frac{1}{10}$
1	RWW or WRW or WWR	$\left(\frac{2}{5} \times \frac{3}{4} \times \frac{2}{3}\right) + \left(\frac{3}{5} \times \frac{2}{4} \times \frac{2}{3}\right) + \left(\frac{3}{5} \times \frac{2}{4} \times \frac{2}{3}\right) = \frac{6}{10}$
2	RRW or RWR or WRR	$\left(\frac{2}{5} \times \frac{1}{4} \times 1\right) + \left(\frac{2}{5} \times \frac{3}{4} \times \frac{1}{3}\right) + \left(\frac{3}{5} \times \frac{2}{4} \times \frac{1}{3}\right) = \frac{3}{10}$

Exercise 11.1

1. Determine if each of the $p(x)$ as described are probability distribution functions. Justify your answer either way.

(a)

x	-3	-2	-1	0
$p(x)$	0.1	0.4	0.4	0.15

(b)

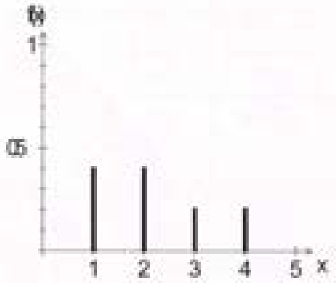
x	-1.5	0	1.5	2.5
$p(x)$	0.5	0.2	0.2	0.1

(c)

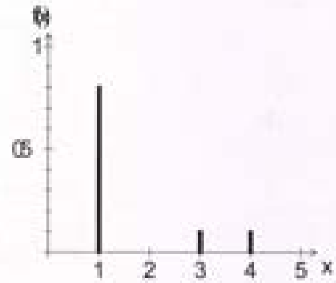
x	1	2	3	4	5	6
$p(x)$	0.1	0.2	0.1	0.3	0.1	0.1

2. Determine if $f(x)$ as described graphically below can be classified as probability distribution functions. Justify your answer in each case.

(a)



(b)



3. Determine if $P(X = x)$ as described below can be classified as a probability distribution for the discrete random variable X. Justify your answer in each case.

(a) $P(X = x) = 1/6$ $\quad X = 1, 2, 3, 4, 5, 6$

(b) $P(X = x) = x/6$ $\quad X = 1, 2, 3$

(c) $P(X = x) = \binom{4}{x} 0.2^x \, 0.8^{4-x}$ $\quad X = 0, 1, 2, 3, 4$

(d) $P(X = x) = 0.8^{x-1} \times 0.2$ $\quad X = 1, 2, 3, 4, 5$ ....

4. The probability distribution for random variable X is tabulated below. Find:

x	0	1	2	3	4
$f(x)$	0.1	0.1	0.2	0.2	0.4

(a) the cumulative probability distribution for X

(b) $P(1 < X \le 3)$ (c) $P(X > 1 \mid X \le 3)$ (d) $P(X \le 3 \mid X > 1)$.

5. The probability distribution function for a discrete random variable X is given by $p(x) = kx$, where $x = 0, 1, 2, 3, 4$. Find:

(a) the value of k (b) the cumulative probability distribution for X

(c) $P(1 < X < 3)$ (d) $P(X > 1 \mid X < 3)$.

6. The probability distribution function for a discrete random variable X is given by $p(x) = \dfrac{k}{x}$, where $x = 1, 2, 3, 4$. Find:

   (a) the value of k
   (b) the cumulative probability distribution for X
   (c) $P(X > 2)$
   (d) $P(X \le 3 \mid X > 2)$.

7. The probability distribution for X is given by $P(X = x) = \dfrac{x^2}{k}$ for $x = -2, -1, 0, 1, 2$.

   Find: (a) the value of k
   (b) the cumulative probability distribution for X
   (c) $P(X > -1)$
   (d) $P(X > -1 \mid X \le 1)$.

8. The probability distribution function of X is tabulated below. Find:

x	0	1	2	3	4
$f(x)$	k	$2k$	$1 - 10k$	$4k$	$3k$

   (a) the value of k
   (b) $P(X \ge 1 \mid X \le 3)$.

9. The probability distribution for X is given by $P(X = x) = \begin{cases} \dfrac{2x+1}{k} & x = 0,1,2,3 \\ \dfrac{11-2x}{k} & x = 4,5 \end{cases}$.

   Find:
   (a) the value(s) of k
   (b) $P(X = 2 \mid 1 \le X \le 4)$

10. The probability distribution for X is given by $P(X = x) = \dfrac{e^{-2}2^x}{x!}$ for $x = 0, 1, 2, 3, \ldots$.

    Find: (a) $P(X \le 2)$ (b) $P(X \le 2 \mid X \le 4)$.

11.3 Mean and Variance of a Discrete Random Variable

- Consider a discrete random variable X with probability distribution function $p(x)$.
  - The mean or expected value of X, μ, is given by:
    $$\mu = E(X) = \sum\left[x \times p(x)\right].$$
  - The variance of X, σ^2, is given by:
    $$\sigma^2 = Var(X) = E(X^2) - [E(X)]^2.$$
    where $E(X^2) = \sum\left[x^2 \times p(x)\right]$
  - The standard deviation of X = $\sqrt{Var(X)} = \sqrt{E(X^2) - [E(X)]^2}$.

Example 11.4

A discrete random variable X has probability distribution function $p(x)$ described in the accompanying table.
Find the mean and standard deviation of X.

x	1	2	3	4
$p(x)$	0.1	0.2	0.4	0.3

Solution:

Mean of X, $E(X) = \sum\left[x \times p(x)\right]$

$= (1 \times 0.1) + (2 \times 0.2) + (3 \times 0.4) + (4 \times 0.3) = 2.9$

$E(X^2) = \sum\left[x^2 \times p(x)\right]$

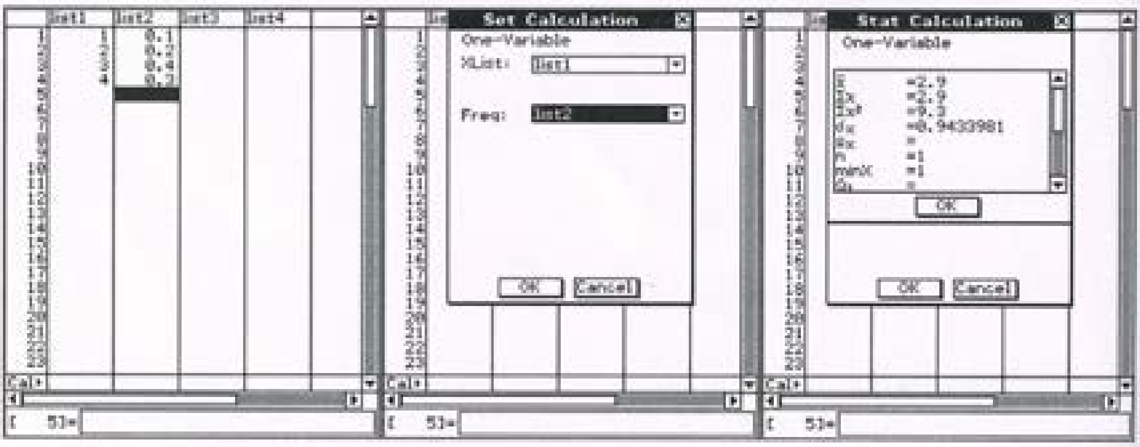
$= (1^2 \times 0.1) + (2^2 \times 0.2) + (3^2 \times 0.4) + (4^2 \times 0.3) = 9.3$

Hence, $Var(X) = E(X^2) - [E(X)]^2 = 9.3 - 2.9^2 = 0.89$

Standard deviation for X = $\sqrt{0.89} = 0.9434$.

Notes:

- *The statistical wizard in CAS/Graphic calculators may be used to calculate the mean and standard deviation for X as shown below. The values of p(x) may are treated as relative frequencies.*



Example 11.5

The probability mass function of a random variable X is given by $p(x) = kx^2$, $x = 1, 2, 3, 4$. Find k. Hence, find the expected value of X and its accompanying variance.

Solution:

Clearly: $p(1) = k$, $p(2) = 4k$, $p(3) = 9k$ and $p(4) = 16k$ and $\sum p(x) = 1$.

But $\sum p(x) = k + 4k + 9k + 16k = 30k$

Hence, $30k = 1 \quad \Rightarrow \quad k = \frac{1}{30}$.

Expected value of X, $E(X) = \sum [x \times p(x)]$

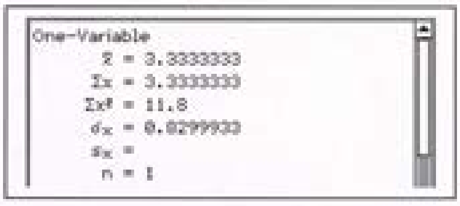
$$= (1 \times k) + (2 \times 4k) + (3 \times 9k) + (4 \times 16k) = 100k = \frac{10}{3}.$$

$$E(X^2) = \sum \left[x^2 \times p(x)\right]$$

$$= (1^2 \times k) + (2^2 \times 4k) + (3^2 \times 9k) + (4^2 \times 16k) = 354k = \frac{59}{5}.$$

$$\Rightarrow \quad Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{59}{5} - \left(\frac{10}{3}\right)^2 = \frac{31}{45}$$



Notes:

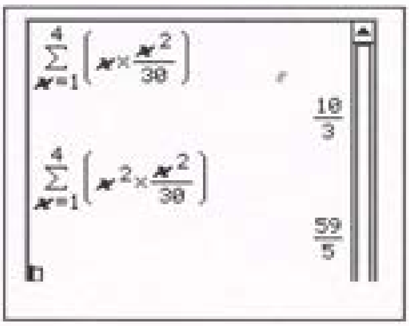
- *The expected value of $X \approx 3.3$. This means that in the long term, the "average value" assumed by X is approximately 3.3.*
- *The spread of the values taken by X is described by its standard deviation. In this case, the standard deviation is approximately 0.83.*
- *The calculations of E(X) and $E(X^2)$ may be done by hand or through the use of the statistical wizard in CAS/graphic calculators. As the "formula" for p(x) is known, we may also use the built in Σ operators found in CAS/graphic calculators as shown below.*

$$E(X) = \sum [x \times p(x)]$$

$$= \sum_{x=1}^{x=4} x \times \frac{x^2}{30} = \frac{10}{3}$$

$$E(X^2) = \sum \left[x^2 \times p(x)\right]$$

$$= \sum_{x=1}^{x=4} x^2 \times \frac{x^2}{30} = \frac{59}{5}$$



Example 11.6

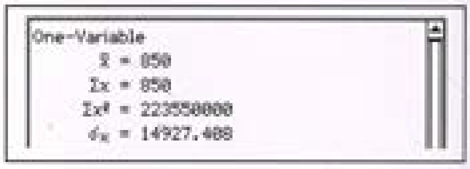
An investment manager developed a risk portfolio with the following returns.

Returns	–$10 000	–$7 000	k	$15 000	$50 000
Probability	0.4	0.2	0.15	0.2	0.05

(a) If k = $5 000, find the expected return for this investment portfolio and the accompanying standard deviation.
(b) Find k if the expected return is $0.

Solution:

(a) Expected Return = $850
Standard deviation ≈ $14 927.



(b) When Expected Return = 0:
$(-10\,000 \times 0.4) + (-7000 \times 0.2) + 0.15k + (15\,000 \times 0.2) + (50\,000 \times 0.05) = 0$
$100 + 0.15k = 0 \;\Rightarrow\; k = -\$666.67$.

Example 11.7

At a fair, a games stall operator offers prizes worth $5, $3, $2 and $1 for each attempt at a particular game. The probabilities of winning these prizes are respectively 0.01, 0.1, 0.1 and 0.7.
(a) Find the probability of not winning a prize.
(b) The games stall operator made a profit of $350 from 200 games. How much did he charge per game?

Solution:

(a) Probability of not winning a prize = 1 – 0.01 – 0.1 – 0.1 – 0.7 = 0.09.

(b) Let the price per game be $$k$.
Let X: Profit per game for the operator.
Hence, the probability distribution for X is:

x	$k-5$	$k-3$	$k-2$	$k-1$	k
$P(X = x)$	0.01	0.1	0.1	0.7	0.09

Given $E(X) = \frac{350}{200} = 1.75$.
But $E(X) = 0.01(k-5) + 0.1(k-3) + 0.1(k-2) + 0.7(k-1) + 0.09k$
Hence, $0.01(k-5) + 0.1(k-3) + 0.1(k-2) + 0.7(k-1) + 0.09k = 1.75$
$\Rightarrow\; k = 3$

Hence, cost per game = $3.

Exercise 11.2

1. Find the mean μ and standard deviation σ associated with each of the following probability distribution functions
   (a) $f(x) = 1/8$ $\quad x = 1, 2, 3, 4, 5, 6, 7, 8$
   (b) $f(x) = x/6$ $\quad x = 1, 2, 3$
   (c) $f(x) = \binom{5}{x} 0.25^x\, 0.75^{5-x}$ $\quad x = 0, 1, 2, 3, 4, 5$
   (d) $P(X = x) = 0.4^{x-1} \times 0.6$ $\quad x = 1, 2, 3, 4, 5, \ldots$

2. The probability distribution for random variable X is tabulated below. Find:

x	1	2	3	4	5
$f(x)$	0.2	0.1	0.1	0.3	0.3

   (a) Find the mean μ and standard deviation σ for X.
   (b) Find $P(X \geq \mu)$. (c) Find $P(\mu - \sigma \leq X \leq \mu + \sigma)$

3. The probability distribution for random variable X is tabulated below. Find:

x	−2	−1	0	1	2	3
$f(x)$	0.1	0.15	0.1	0.2	0.2	0.25

   (a) Find the mean μ and standard deviation σ for X.
   (b) Find $P(X \geq \mu)$. (c) Find $P(\mu - \sigma \leq X \leq \mu + \sigma)$

4. The probability distribution function for a discrete random variable X is given by $p(x) = kx^2$, where $x = 0, 1, 2, 3, 4, 5, 6, 7$. Find:
   (a) the value of k (b) the mean and standard deviation for X.

*5. The probability mass function for X is given by $P(X = x) = \dfrac{e^{-5}5^x}{x!}$ for $x = 0, 1, 2, 3, \ldots$.
   (a) Find the expected value and standard deviation for X.
   (b) Find the most likely value for X.

6. The probability mass function for X is given by $P(X = x) = \dfrac{\binom{6}{x}\binom{7}{4-x}}{\binom{13}{4}}$ for $x = 0, 1, 2, 3, 4$.
   (a) Find the expected value and standard deviation for X.
   (b) Find the most likely value for X.

7. The probability distribution function of X is tabulated below. Given that E(X) = 2, find:

x	0	1	2	3	4	5
$f(x)$	0.2	a	0.1	0.2	b	0.1

   (a) the values of a and b (b) $P(X \leq 1 \text{ or } X \geq 4)$.

8. The probability distribution function of X is tabulated below. Given that E(X) = 1, find:

x	-2	a	b	1	3	7
$f(x)$	$0.1b$	0.1	0.2	$0.2a$	0.1	0.1

(a) the values of a and b (b) k if $P(X \le k) = 0.7$.

*9. The probability mass function of X is given by $p(x) = ax$ for $x = 1, 2, 3, 4, \dots, k$.
The expected value of X is 7. Find a and k.

*10. The probability mass function of X is given by $p(x) = ax^3$ for $x = 1, 2, 3, \dots, k$.
The expected value of X is 177/50. Find a and k.

11. An investment manager, after some research developed the following risk portfolio with five considered options and their respective returns.

Returns	−\$5 000	−\$3 000	b	\$5 000	\$30 000
Probability	0.3	a	0.15	0.15	0.05

(a) Find a. (b) Find the expected return if b = \$2000.
(c) Find b if the expected return is \$900. (d) Find b if the expected return is a loss.

12. An investment manager, after some research developed the following risk portfolio with five considered options and their respective returns.

Returns	−\$10 000	a	b	\$5 000	\$20 000
Probability	0.05	0.4	0.3	0.2	0.05

(a) Find the expected return in terms of a and b.
(b) Given that $a + b = 0$ and that the expected return is \$0, find a and b.
(c) Find a in terms of b if the portfolio must return a profit.

13. Noel invests \$100 000 in a scheme which lists the value of the initial investment after five years as follows.

Value	k	\$160 000	\$110 000	\$75 000	\$50 000
Probability	0.1	0.4	0.3	0.15	0.05

(a) For k = \$200 000, find the expected value of the initial investment after five years.
(b) Find the value for k if the expected value after five years is \$100 000.

14. At a country fair, a games stall operator offers prizes worth \$10, \$5, \$2 and \$1 for one attempt at a particular game. The probabilities of winning these prizes are respectively 0.005, 0.01, 0.1 and 0.3.
(a) Find the probability of not winning a prize.
(b) If each game costs \$1, find the expected profit per game for the games stall operator.
(c) Julia paid \$2 for a game. Find her expected profit/loss.
(d) The games stall operator made a profit of \$400 from 200 games. How much did he charge per game?

15. At an agricultural fair, a games stall operator offers prizes worth \$1.50, \$2, \$1 and \$0.50 for one attempt at a particular game. The probabilities of winning these prizes are respectively 0.1, 0.01, 0.1 and 0.3.
    (a) Find the probability of not winning a prize.
    (b) If each game costs \$2, find the expected profit per game for the games stall operator.
    (c) If Julius paid \$1 for a game, find his expected profit/loss.
    (d) The games stall operator made a profit of \$540 from 500 games. How much did he charge per game?

16. Two fair coins are tossed. Let X: number of heads obtained.
    (a) Describe the probability distribution and cumulative probability distribution for X.
    (b) Find the most likely value for X and the mean value for X.

17. Two regular fair die are rolled. Let X: absolute value difference between the scores on the two die.
    (a) Describe the probability distribution and cumulative probability distribution for X.
    (b) Find the most likely value for X.

18. A box contains 4 green discs and 3 yellow disks. Three discs are drawn randomly from the box without replacement. Let X: number of green discs drawn.
    (a) Describe the probability distribution and cumulative probability distribution for X.
    (b) Find the most likely value for X and the mean value for X and its variance.

19. An urn contains 4 black discs and 2 white discs. Three discs are drawn randomly from the box with replacement. Let X: number of white discs drawn. What is the most likely number of white discs and the mean number of white discs that will be drawn? Justify your answer.

20. A fair coin is tossed. If a head appears, a normal fair six-sided die is tossed. Otherwise, a fair 8-sided die (marked 1 through to 8) is tossed. Let X: number obtained from the die.
    (a) Find $P(X \le 7 \mid X \ge 3)$.
    (b) Find the mean, standard deviation and the most likely value for X.

21. In an experiment, a child is required to randomly select, without replacement, 3 balls from a box containing 8 red and 2 green balls. For every red ball chosen, the child is rewarded with 1 box of chocolates and for every green ball chosen; the child is rewarded with 5 boxes of chocolates.
    (a) Define the random variable X as the number of green balls selected. Find the probability distribution for X.
    (b) Define the random variable T as the number of boxes of chocolates given to the child. Find the probability distribution for T. Hence, determine the most likely number and mean number of boxes of chocolates given to the child.

22. In the game of Lotto, six numbers are randomly drawn from the numbers 1 through to 45. Players are required to guess which six numbers will be drawn by marking their guesses on an entry coupon. Determine with reasons the most likely number of correct guesses?

23. A student is required to play over the school FM radio station a classical piece of music followed immediately by a contemporary piece of music. To choose between classical piece A and classical piece B, each with playing times 12 minutes and 11 minutes respectively, the student tosses a fair coin. If a head turns up, then piece A is played, otherwise piece B is played. To choose between contemporary pieces C and D, each with playing times 2 minutes and 3 minutes respectively, a fair die is tossed. If a number less than 3 is obtained, then C is played, otherwise D is played. Let T: playing time for the 2 pieces. Find the probability distribution for T.
List in decreasing order of likelihood the combination of pieces that will be played.

24. In a quiz show, each contestant in turn is required to answer a series of questions. Each correct response will be followed by another question up to a maximum of five questions. A contestant is declared "out" if a wrong answer is given to any question and gets no more questions. The number of points scored is equal to the number of questions answered correctly except in the case where 5 questions have been answered correctly whereby the contestant is awarded a bonus point. Mark has a probability of 0.9 of answering any question correctly. Assume that each response is independent of the others. Find the probability distribution for T, the total number of points scored by Mark. What is Mark's most likely score and mean score for this quiz.

25. A cereal manufacturer inserts a "cartoon" card in 1% of its 500g breakfast cereal line. Let N: Number of boxes of breakfast cereal, Amy has to open before she gets the first "cartoon" card.
   (a) Find the probability distribution for N.
   (b) Find the mean number of boxes she has to open before she gets the first "cartoon" card.

26. Two fair dice are thrown and the numbers A and B shown on each dice are noted.

   The score X from the throw is defined by $X = \begin{cases} A+B & \text{if } A=B \\ A-B & \text{if } A>B \\ B-A & \text{if } A<B \end{cases}$

   (a) Determine the probability distribution for X.
   (b) What is the most likely value of X that will occur?
   (c) What is the mean value of X.

27. The random variable X has probability distribution given by $P(X = x) = \frac{x}{10}$,

   for $x = 1, 2, 3, 4$.
   (a) Find E(X) and Var(X).
   (b) Let the random variable F = 2X – 3.
   Find the probability distribution for F and hence E(F) and Var(F).
   (c) Let the random variable G = aX + b, where a and b are constants.
   Find the probability distribution for G and hence E(G) and Var(G).
   (d) Use your answers in (b) and (c) to determine the relationship between E(G) and E(X), and the relationship between Var(G) and Var(X).

11.4 Discrete Uniform Distributions

- A discrete random variable X with probability mass function of the form

$$p(x) = \frac{1}{n} \quad \text{for } x = 1, 2, 3, \ldots, n$$

is called a discrete uniform variable or a discrete rectangular variable.

  - Clearly for a discrete uniform variable,

$$p(1) = p(1) = p(1) = \ldots = p(n) = \frac{1}{n} \quad \text{(constant for all } x\text{)}.$$

  - The mean and variance for X are respectively $\frac{n+1}{2}$ and $\frac{n^2-1}{12}$.

- For X as a discrete uniform variable, the outcomes for X are each equally likely. That is, any discrete variable with equally likely outcomes may be classed as a discrete uniform variable.
  Examples of discrete uniform variables are:
  - X: the outcome of the toss of a fair n-sided die
  - X: the outcome of the draw of a single ball from a barrel with n distinctly numbered balls

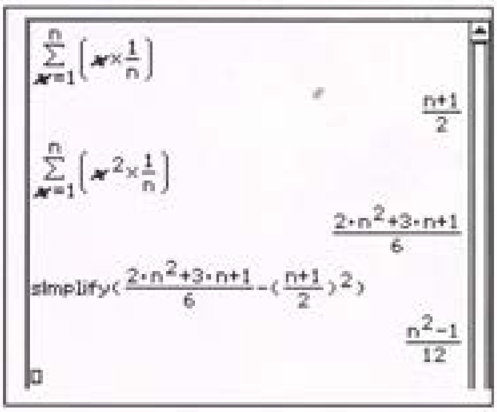
Example 11.8

The probability mass function for a discrete uniform variable is given by $p(x) = \frac{1}{n}$ for $x = 1, 2, 3, \ldots, n$. Find the expected value and variance for X.

Solution:

The expected value for X:

$$\mu = \sum_{x=1}^{x=n}\left(x \times \frac{1}{n}\right) = \frac{n+1}{2}.$$



The variance for X:

$$\begin{aligned}\sigma^2 &= E(X^2) - [E(X)]^2 \\ &= E(X^2) - \mu^2 \\ &= \sum_{x=1}^{x=n}\left(x^2 \times \frac{1}{n}\right) - \left(\frac{n+1}{2}\right)^2 \\ &= \frac{2n^2+3n+1}{6} - \left(\frac{n+1}{2}\right)^2 \\ &= \frac{n^2-1}{12}.\end{aligned}$$

Exercise 11.3

1. A discrete random variable X where X = 0, 1, 2, 3, 4 has a uniform distribution. Find the probability distribution function, expected value and variance for X.

2. A discrete random variable X, where X = −2, −1, 0, 1, 2, 3, 4; has a uniform distribution. Find the mean and standard deviation for X.

3. The probability mass function for a uniform discrete variable is given by $p(x) = \frac{1}{n+1}$ for $x = 0, 1, 2, 3, \ldots, n$. Find the expected value and variance for X.

4. A fair normal six-sided die is rolled. Define X: Number of dots on the uppermost face of the die. Find the expected value for X and its associated standard deviation.

5. The faces of a fair eight-sided die are numbered 1 through to 8 inclusive. The die is rolled. Define X: Number label on the uppermost face of the die. Find the expected value for X and its associated standard deviation.

6. A charged particle has an equally likely chance of being in any one of the following states. Define X: State of the charged particle where the values of x are as defined in the table below.

x	−2	−1	0	1	2	3
State	Negative with clockwise spin	Negative with anti-clockwise spin	Negative with no spin	Positive with no spin	Positive with anti-clockwise spin	Positive with clockwise spin

   (a) Find the probability mass distribution for X. State its mean and standard deviation.
   (b) Interpret: (i) $P(1 \le X \le 3)$ (ii) $P(X = -2 \cup X = 3)$.
   (c) Find the probability that a charged particle with spin has a positive charge

7. A random digit generator generates numbers randomly between 000 and 999 inclusive. Let X: Number generated by a random digit generator.
   (a) Find the probability distribution for X.
   (b) Find μ, the expected number generated and its associated standard deviation.
   (c) Find the probability that a number generated greater than μ is not more than $\mu + \sigma$.

8. A discrete random variable X where X = 1, 2, 3, 4, … n; has a uniform distribution. It is known that E(X) = 11. Find:
   (a) the probability distribution function for X (b) the variance for X.

9. A discrete random variable X where X = 0, 1, 2, 3, 4, … n; has a uniform distribution. It is known that E(X) = 5. Find:
   (a) the probability distribution function for X (b) the variance for X.

10. A random digit generator generates numbers randomly between 0000 and n inclusive. Let X: Number generated by a random digit generator. Find n if the variance for X is 1 334 000.

11.5 Linear changes on Random Variables.

- In this section, we will examine the effect on the mean and variance of a random variable when the random variable undergoes a linear transformation.

Hands On Task 11.1

In this task, we will examine the mean and variance of a transformed variable.

Consider the random variable X with probability distribution function $p(x) = \frac{x}{10}$ for $x = 1, 2, 3, 4$. The mean for X is given by $E(X) = \sum[x \times p(x)]$.
The variance for X is given by $Var(X) = E(X^2) - [E(X)]^2$.

$$\sum_{x=1}^{4}\left(x \times \frac{x}{10}\right)$$

$$\sum_{x=1}^{4}\left(x^2 \times \frac{x}{10}\right) - \left(\sum_{x=1}^{4}\left(x^2 \times \frac{x}{10}\right)\right)^2$$

1. Calculate:
   (a) $E(X) = \sum[x \times p(x)]$.
   (b) $Var(X) = \sum\left[x^2 \times p(x)\right] - [E(X)]^2$

2. Let Y = 3X. Clearly the range for Y = 3, 6, 9, 12.
   Also: $P(Y = 3) \equiv P(X = 1) = \frac{1}{10}$, $P(Y = 6) \equiv P(X = 2) = \frac{2}{10}$
   $P(Y \equiv 9) = P(X = 3) = \frac{3}{10}$, $P(Y = 12) \equiv P(X = 4) = \frac{4}{10}$
   That is, the probability distribution function for Y is $p(y) = \frac{y}{30}$ for $y = 3, 6, 9, 12$.
   (a) Verify that E(Y) = 3 E(X). (b) Verify that Var(Y) = 9 Var(X).

3. Let Y = 3X – 4.
   (a) Find the probability distribution for Y. (b) Verify that E(Y) = 3E(X) – 4.
   (c) Verify that Var(Y) = 9 Var(X).

4. Let Y = –5X + 2.
   (a) Find the probability distribution for Y. (b) Verify that E(Y) = –5E(X) + 2.
   (c) Verify that Var(Y) = 25 Var(X).

5. Let Y = aX + b where a and b are constants.
   (a) Verify that $E(aX + b) = aE(X) + b$.
   (b) Verify that $Var(Y) = a^2 Var(X)$

11.5.1 Effects of linear changes on the mean and standard deviation.

- Let the random variable X have probability distribution $P(X = x)$ with mean μ and variance σ^2. Let $Y = aX + b$ where a and b are constants.

- The following results are quoted without proof (the proof of which is beyond the scope of this book).
  - The probability distribution for Y is $P(Y = ax + b) \equiv P(X = x)$.
  - The mean for $Y = a\mu + b$.
  - The variance for $Y = a^2\sigma^2$.
  - The standard deviation for $Y = |a|\sigma$.

- Note that the standard deviation is affected by the "change of scale factor a" but not affected by the "change of origin factor b".

- The mean however is affected by both the "change of scale factor a" and the "change of origin factor b".

Example 11.9

The probability mass function for a random variable is given by

$p(x) = \dfrac{x+1}{10}$ for $x = 0, 1, 2, 3$. Given that $Y = -2X + 3$, find:

(a) $P(Y = -1)$ (b) $P(Y \geq 0)$ (c) the mean and standard deviation for Y.

Solution:

(a) $Y = -1 \Rightarrow -2X + 3 = -1 \Rightarrow X = 2$

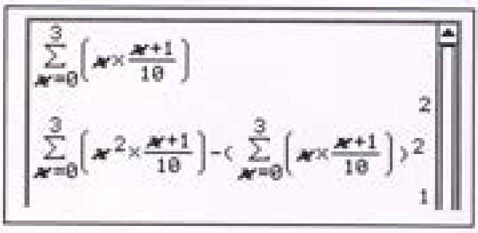
$P(Y = -1) \equiv P(X = 2) = \dfrac{3}{10}$

(b) $Y \geq 0 \Rightarrow -2X + 3 \geq 0 \Rightarrow X \leq 1.5$

$P(Y \geq 0) \equiv P(X = 0) + P(X = 1) = \dfrac{3}{10}$.

(c) Mean for $X = \displaystyle\sum_{x=0}^{3}\left(x \times \frac{x+1}{10}\right) = 2$

Var for $X = \displaystyle\sum_{x=0}^{3}\left(x^2 \times \frac{x+1}{10}\right) - 2^2 = 1$



Hence, mean for $Y = -2 \times 2 + 3 = -1$.

Standard deviation for $Y = |-2| \times 1 = 2$

Example 11.10

The random variable X has mean 40 and standard deviation 8. The random variable $Y = aX + b$. Find a and b if the mean and standard deviation for Y are 15 and 2 respectively

Solution:

Mean for Y = a × (Mean for X) + b $\Rightarrow$ $15 = 40a + b$
Standard deviation for Y = $|a|$ × Standard deviation for X

$$\Rightarrow \quad 2 = |a| \times 8$$

$$|a| = \frac{1}{4}$$

If $a = \frac{1}{4} \Rightarrow b = 5$. If $a = -\frac{1}{4} \Rightarrow b = 25$

Hence, $a = \frac{1}{4}, b = 5$ or $a = -\frac{1}{4}, b = 25$

Example 11.11

The random variable X has mean μ and standard deviation σ. Show that the random variable $Y = \frac{X-\mu}{\sigma}$ will have a mean of 0 and a standard deviation of 1.

Solution:

$$Y = \frac{X-\mu}{\sigma} \Rightarrow Y = \frac{1}{\sigma}X - \frac{\mu}{\sigma}$$

$$\text{Mean for Y} = \frac{1}{\sigma} \times (\text{Mean for X}) - \frac{\mu}{\sigma}$$

$$= \frac{1}{\sigma} \times \mu - \frac{\mu}{\sigma} = 0$$

$$\text{Standard deviation for Y} = \frac{1}{\sigma} \times \text{Standard deviation for X}$$

$$= \frac{1}{\sigma} \times \sigma = 1$$

The Standard Score Transform

- The transform $\frac{X-\mu}{\sigma}$ is an important transform.
  It is often referred to as the *standard score* transform.
  It transforms *all* random variables into random variables with mean 0 and standard deviation 1.

Exercise 11.4

1. The probability distribution function for a random variable X is given by $p(x) = \frac{1}{6}$ for $x = 1, 2, 3, 4, 5, 6$. Given that random variable $Y = 2X + 1$, find the:
   (a) probability distribution function for Y. (b) mean and standard deviation for Y.

2. The probability distribution function for a random variable X is given by $p(x) = \frac{x}{15}$ for $x = 1, 2, 3, 4, 5$. Given that random variable $Y = 4 - X$, find the:
   (a) probability distribution function for Y. (b) mean and standard deviation for Y.

3. The probability distribution function for a random variable X is given by $p(x) = \frac{x+2}{20}$ for $x = 0, 1, 2, 3, 4$. Given that random variable $Y = \frac{X+2}{60}$, find:
   (a) $P(Y = \frac{1}{15})$ and $P(Y = \frac{1}{15} \mid Y \geq \frac{1}{20})$. (b) the mean and standard deviation for Y.

4. The probability distribution function for a random variable X is given by $p(x) = \binom{4}{x} 0.4^x \, 0.6^{4-x}$ for $x = 0, 1, 2, 3, 4$. Given that random variable $Y = 5X + 10$, find:
   (a) $P(15 \leq Y \leq 25)$ and $P(Y \leq 25 \mid Y \geq 15)$. (b) the mean and standard deviation for Y.

5. The probability distribution function for a random variable X is given by
   $p(x) = \frac{\binom{6}{x}\binom{4}{4-x}}{\binom{10}{4}}$ for $x = 0, 1, 2, 3, 4$. Given that random variable $Y = 4 - 2X$, find:
   (a) $P(Y \leq 2)$ and $P(Y \leq 0 \mid Y \leq 2)$. (b) the mean and standard deviation for Y.

6. A discrete random variable X, where $X = -3, -2, -1, 0, 1, 2, 3$ has a uniform distribution.
   (a) Find the mean and variance for $Y = 6 - X$.
   (b) Find the mean and variance for $W = 2X + Y$

7. The random variable X has mean -20 and standard deviation 4. The random variable $Y = aX + b$. Find a and b if the mean and standard deviation for Y are 40 and 12 respectively.

8. The random variable X has mean 80 and standard deviation 30. The random variable $Y = aX + b$. Find a and b if the mean and standard deviation for Y are 60 and 15 respectively.

9. The mean and standard deviation for a random variable X are 1.54 metres and 0.32 metres respectively. Find the mean and standard deviation if each value of X is:
   (a) increased by 0.5 metres
   (b) increased by 10%
   (c) decreased by 15%
   (d) halved and then added to 0.9 metres.

10. The mean and standard deviation for random variable X are μ and σ respectively.
    (a) Random variable Q is formed by adding 10 to each of the values of X. Find the values of μ and σ if the mean and standard deviation for random variable Q are 46 and 3.5 respectively.
    (b) Random variable R is formed by decreasing each of the values of X by 25%. Find the values of μ and σ if the mean and standard deviation for random variable R are 61 and 5.4 respectively.
    (c) Random variable S is formed by subtracting 5 from each of the values of X and then increasing each of the results by 20%. Find the values of μ and σ if the mean and standard deviation for random variable S are 46 and 3.5 respectively.

11. The random variable X has a mean and standard deviation of 24 and 3.2 respectively.
    (a) Find the standard score corresponding to an initial score of 35 and 8.
    (b) Find the initial score that corresponds to a standard score of −2.5 and 1.9

12. The mean and standard deviation for random variable X are μ and 12 respectively. Find the value of μ if the standard score that corresponds to a score of 48 is −0.5.

13. The mean and standard deviation for a random variable Y are 124 and σ respectively. Find the value of σ if the standard score that corresponds to a score of 136 is 1.5.

14. The mean and standard deviation for a random variable W are μ and σ respectively. The standard score that corresponds to a score of 52 is 2 while the standard score that corresponds to a score of 27 is −3. Find μ and σ.

15. The mean and standard deviation for a Science Competition Paper (marked out of 100) in 2013 was 67 and 23 respectively. In 2014, the mean and standard deviation was 54 and 15 respectively. Jamie scored 62 and 52 in the 2013 and 2014 Competitions respectively. For purposes of comparison, the marks for each of the two years were converted to have means of 50 and standard deviations of 16.6.
    (a) Find the standard scores for Jamie's scores in the 2013 and 2014 competitions.
    (b) Find Jamie's converted scores in the 2013 and 2014 competitions and hence determine which is the better of Jamie's two performances.

*16. The mean and standard deviation for random variable X is 68 and 15 respectively. The mean and standard deviation for random variable Y is 61 and 17 respectively. Use standard scores to determine if $P(X \geq 72)$ is more likely to be greater than, equal to or less than $P(Y \geq 65)$.

12 The Binomial Distribution

12.1 Bernoulli Variables

- Consider the statistical experiment where a fair coin is tossed once.
  - The sample space consists of exactly two outcomes; { Head, Tail }.
  - Define the event of obtaining a "Head" as a successful outcome.
    Further, define $X = \begin{cases} 1 & \text{if outcome is a Head (success)} \\ 0 & \text{if outcome is a Tail (failure)} \end{cases}$
  - Then, $p = \text{P(success)} = \text{P}(X = 1) = 0.5$.
  - X is termed a Bernoulli variable with parameter $p = 0.5$.
    The mean for X, $\mu = \text{E}(X) = 0.5$.
    The variance for X, $\sigma^2 = \text{E}(X^2) - \mu^2 = 0.5 - 0.5^2 = 0.25$.
- Consider the statistical experiment where a fair six-sided die is tossed once.
  - The sample space consists of exactly six outcomes; {1, 2, 3, 4, 5, 6}.
  - Define the event of obtaining the "number 1" as a successful outcome.
    In this instance, the sample space may be rewritten as {1, "not 1"}
    Further, define $X = \begin{cases} 1 & \text{if outcome is a One (success)} \\ 0 & \text{if outcome is not a One (failure)} \end{cases}$
  - Then, $p = \text{P(success)} = \text{P}(X = 1) = \frac{1}{6}$.
  - X is termed a Bernoulli variable with parameter $p = \frac{1}{6}$.
    The mean for X, $\mu = \text{E}(X) = \frac{1}{6}$.
    The variance for X, $\sigma^2 = \text{E}(X^2) - \mu^2 = \frac{1}{6} - \left(\frac{1}{6}\right)^2 = \frac{5}{36}$.
- In general, a Bernoulli Trial is a statistical experiment where the outcomes may be grouped into a "success" category and a "failure" category.
- A Bernoulli variable assigns the "success" category the numerical value of 1 and the "failure" category the numerical value of 0.
- If X is a Bernoulli variable with parameter p, (p is the probability of success) then:
  - the probability mass function of X is $p(x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \end{cases}$
  - the mean for X, $\mu = \text{E}(X) = p$
    the variance for X, $\sigma^2 = \text{E}(X^2) - \mu^2$
    $= p - p^2$
    $= p(1 - p)$
    $= pq \qquad$ where $q = \text{P(Failure)} = 1 - p$.

Example 12.1

It is known that 15% of students at a certain school are first born (first child in the family). A student is randomly chosen from this school. Define X = 1 if the student chosen is a first born and X = 0 otherwise. Find the expected value of X and its associated standard deviation.

Solution:

Clearly X is a Bernoulli variable.
Hence, expected value of X = 0.15.
Variance = 0.15 × 0.85 $\Rightarrow$ Standard deviation = $\sqrt{0.15\times0.85}$ = 0.3571

12.2 The Binomial Random Variable

- Consider the Bernoulli Trial where a fair six sided die is rolled once and where success is achieved when a "one" appears.
  - Define the random variable X as:

    $$X = \begin{cases} 1 & \text{if outcome is a One (success)} \\ 0 & \text{if outcome is not a One (failure)} \end{cases}$$

  - Clearly X is a Bernoulli Variable with parameter $p = \frac{1}{6}$

    and $E(X) = \frac{1}{6}$ and $Var(X) = \frac{5}{36}$.
- Let the Bernoulli Trial described above be repeated n times. That is; the die is rolled n times and for each trial, success is achieved when a "one" appears.
  - Let X_i for $i = 1$ to n represent the n Bernoulli Trials.
  - Define Y: Number of successful trials from the n Bernoulli Trials.

    Since, $X_i = \begin{cases} 1 & \text{if outcome is a One (success)} \\ 0 & \text{if outcome is not a One (failure)} \end{cases}$ for $i = 1$ to n,

    Clearly, $Y = X_1 + X_2 + X_3 + \ldots + X_n$.

    That is, Y is the sum of n Bernoulli variables each with parameter $p = \frac{1}{6}$.
  - $P(Y = r) = P(r$ successes and $n - r$ failures)
    - In n trials, there are $\binom{n}{r}$ possible arrangements of r successes and $n - r$ failures.
    - The probability of each such arrangement is $p^r(1-p)^{n-r}$.
    - Hence, $P(Y = r) = P(r$ successes and $n - r$ failures)

      $$= \binom{n}{r} p^r (1-p)^{n-r} \quad \text{for } r = 1, 2, 3, \ldots, n.$$
  - Note that $P(Y = r)$ are the terms of the binomial expansion $(p + (1-p))^n$.
  - The random variable Y is termed a Binomial Variable.

- The expected value of Y,

$$\begin{aligned} E(Y) &= E(X_1 + X_2 + X_3 + \ldots + X_n) \\ &= E(X_1) + E(X_2) + E(X_3) + \ldots + E(X_n) \qquad \text{I} \\ &= p + p + p + \ldots + p \\ &= np \end{aligned}$$

- The variance of Y,

$$\begin{aligned} Var(Y) &= Var(X_1 + X_2 + X_3 + \ldots + X_n) \\ &= Var(X_1) + Var(X_2) + Var(X_3) + \ldots + Var(X_n) \qquad \text{II} \\ &= pq + pq + pq + \ldots + pq \\ &= npq \end{aligned}$$

- The proofs of the results I and II are beyond the scope of this unit.

- In general, if:

  X: Number of successes in n *independent* Bernoulli Trials where P(success) = p
  - then, X is called a Binomial Random Variable with parameters n and p.

    In symbols, $X \sim B(n, p)$.
  - The probability distribution for X is given by:

$$P(X = r) = \binom{n}{r} p^r (1-p)^{n-r} \quad \text{where } r = 0, 1, 2, 3 \ldots, n.$$

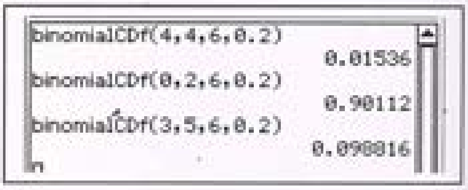
- The expected value for X, or the mean for X, $E(X) = np$.
- The variance for X, $Var(X) = npq$ where $q = 1 - p$.
- The probability values are calculated through the use of the above formula. However, CAS/Graphic calculators have built-in routines and programmes that generate these probability values.

Example 12.2

Given that $X \sim B(6, 0.2)$, find: (a) $P(X = 4)$ (b) $P(X \le 2)$ (c) $P(3 \le X \le 5)$

Solution:

(a) $P(X = 4) = 0.0154$



(b) $P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2)$
$= P(0 \le X \le 2) = 0.9011$

(c) $P(3 \le X \le 5) = P(X = 3) + P(X = 4) + P(X = 5)$
$= 0.0988$

Example 12.3

Given that X ~ B(4, 0.2), find, without the use of a calculator: (a) P(X = 3) (b) P(X ≤ 1).

Solution:

(a) $P(X=3)=\binom{4}{3}\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^1 = 4\times\frac{1}{125}\times\frac{4}{5}=\frac{16}{625}$.

(b) $P(X\le 1) = P(X=0)+P(X=1)$

$$=\left(\frac{4}{5}\right)^4+\binom{4}{1}\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)^3$$

$$=\frac{256}{625}+\frac{256}{625}=\frac{512}{625}.$$

Example 12.4

Given that X ~ B(10, 0.65), find:

(a) P(X ≥ 7) (b) P(X < 9) (c) P(7 ≤ X < 9) (d) P(X ≥ 7 | X < 9)

(e) the mean value and most likely value of X

Solution:

(a) $P(X\ge 7) = P(7\le X\le 10)$
$= 0.51383 = 0.5138$

(b) $P(X<9) = P(0\le X\le 8) = 0.91405 = 0.9141$

(c) $P(7\le X<9) = P(7\le X\le 8)$
$= 0.42787 = 0.4279$

(d) Using the Conditional rule for probability:

$$P(X\ge 7\mid X<9)=\frac{P(X\ge 7\cap X<9)}{P(X<9)}$$

$$=\frac{P(7\le X\le 8)}{P(0\le X\le 8)}$$

$$=\frac{0.42787}{0.91405}=0.4681$$

```
binomialCDf(7,10,10,0.65)
                          0.51383
binomialCDf(0,8,10,0.65)
                          0.91405
binomialCDf(7,8,10,0.65)
                          0.42787
0.42787
0.91405
                          0.46810
binomialCDf(6,6,10,0.65)
                          0.23767
binomialCDf(7,7,10,0.65)
                          0.25222
binomialCDf(8,8,10,0.65)
                          0.17565
```

(e) Mean value for X = 10 × 0.65 = 6.5
P(X = 6) = 0.23767, P(X = 7) = 0.25222 and P(X = 8) = 0.17565
Hence, most likely value for X is 7.

Note:

- *The most likely value of X is the value of X with the highest probability. This value is usually close to the mean value of X.*

Example 12.5

Given that X ~ B(11, p), where $0 < p < 1$, state the probability distribution function for X.
Find: (a) P(X ≤ 1) (b) P(X ≤ 10) (c) P(2 ≤ X < 4) (d) E(X) and Var(X).

Solution:

The probability distribution for X is given by:

$$P(X = r) = \binom{11}{r} p^{r} (1-p)^{n-r}, \text{ where } r = 0, 1, 2, 3, \ldots, 11.$$

(a) $P(X \le 1) = P(X = 0) + P(X = 1)$

$$= \binom{11}{0} p^{0} (1-p)^{n} + \binom{11}{1} p^{1} (1-p)^{n-1}$$

$$= (1-p)^{n} + 11p(1-p)^{n-1} = (1-p)^{n-1}[1 + 10p]$$

(b) $P(X \le 10) = 1 - P(X = 11)$

$$= 1 - \binom{11}{11} p^{11} (1-p)^{0} = 1 - p^{11}$$

(c) $P(2 \le X < 4) = P(X = 2) + P(X = 3)$

$$= \binom{11}{2} p^{2} (1-p)^{9} + \binom{11}{3} p^{3} (1-p)^{8}$$

$$= 55p^{2}(1-p)^{9} + 165p^{3}(1-p)^{8} = 55p^{2}(1-p)^{8}(1+2p)$$

(d) $E(X) = 11p$ and $Var(X) = 11p(1-p)$.

Example 12.6

Given that X ~ B(n, 0.01), find the maximum value for n such that P(X > 1) > 0. 1.

Solution:

It is required that P(X > 1) > 0.1.

Rewriting $1 - P(X = 0) - P(X = 1) > 0.1$

$P(X = 0) + P(X = 1) < 0.9$

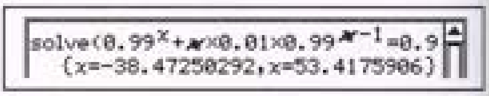
Therefore $\binom{n}{0} 0.99^{n} + \binom{n}{1} 0.01 \times 0.99^{n-1} < 0.9$

$0.99^{n} + n \times 0.01 \times 0.99^{n-1} < 0.9$

For $0.99^{n} + n \times 0.01 \times 0.99^{n-1} = 0.9$

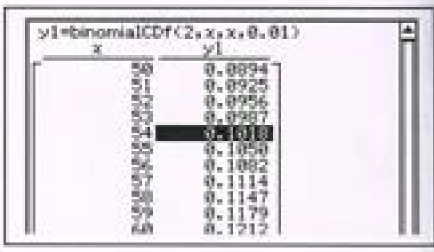
$n = 53.4$

Hence, $n \ge 54$



Alternative Solution:

P(X > 1) > 0. 1 ⇒ P(X ≥ 2) > 0.1
Use CAS Calculator:
binomialCDf(2, x, x, 0.01) > 0.1 ⇒ $n \ge 54$.



Exercise 12.1

1. Given that $X \sim B(8, 0.4)$, find:
   (a) $P(X = 1)$ (b) $P(X \le 5)$ (c) $P(X \ge 1)$ (d) $P(X \ge 1 \mid X \le 5)$
   (e) $P(X \le 5 \mid X \ge 1)$ (f) the mean value and most likely value for X.

2. Given that $X \sim B(8, 0.45)$, find:
   (a) $P(X = 1)$ (b) $P(X \le 3)$ (c) $P(X \ge 1)$ (d) $P(X \ge 1 \mid X \le 3)$
   (e) $P(X \le 3 \mid X \ge 1)$ (f) the mean value and most likely value for X.

3. Given that $X \sim B(4, p)$, where $0 \le p \le 1$, find:
   (a) $P(X = 1)$ (b) $P(X \le 1)$ (c) $P(X \ge 3)$ (d) $E(X)$ and $Var(X)$.

4. Given that $X \sim B(9, p)$, where $0 \le p \le 1$, find:
   (a) $P(X = 4)$ (b) $P(X \le 1)$ (c) $P(X > 8)$ (d) $E(X)$ and $Var(X)$.

5. Given that $X \sim B(n, 0.2)$, where n is a positive integer, find:
   (a) $P(X = 3)$ (b) $P(X < 1)$ (c) $P(X > n - 1)$ (d) $E(X)$ and $Var(X)$.

6. Given that $X \sim B(10, 0.35)$, and $2 \le k \le 4$, find:
   (a) $P(X = k)$ (b) $P(k < X < k + 2)$ (c) $P(k - 1 < X < k + 1)$

7. Given that $X \sim B(n, 0.2)$, find n given that $P(X = 0) = 0.06872$.

8. Given that $X \sim B(n, 0.9)$, find n given that $P(X = n) = 0.04239$.

9. Given that $X \sim B(n, 0.2)$, find the largest value of n for which:
   (a) $P(X \ge 1) \le 0.75$ (b) $P(X > 1) \le 0.4$.

10. Given that $X \sim B(n, 0.01)$, find the largest value of n for which:
    (a) $P(X \ge 1) \le 0.6$ (b) $P(X > 1) \le 0.1$.

11. Given that $X \sim B(n, 0.1)$, find the smallest value of n for which:
    (a) $P(X \ge 1) \ge 0.7$ (b) $P(X \ge n - 1) \le 0.01$.

12. Given that $X \sim B(n, 0.12)$, find the smallest value of n for which:
    (a) $P(X \ge 1) \ge 0.95$ (b) $P(X > n - 2) \le 0.0001$.

13. Given that $X \sim B(n, p)$, find n and p if $E(X) = 23$ and $Var(X) = 17.71$.

14. Given that $X \sim B(n, p)$, find n and p if the mean and standard deviation for X are
    (a) 28 and $\frac{2\sqrt{105}}{5}$ respectively. (b) 20 and $\frac{2\sqrt{210}}{7}$ respectively.

Example 12.7

Assume that 15% of students at a certain school are left-handed. Ten students are randomly selected from this school. Define X: Number of left-handed students selected.
Show clearly that X is a binomial variable. State the probability mass function, expected value and variance for X.

Solution:

X: Number of left-handed students out of the 10 students selected.

Let $Y_i = \begin{cases} 1 & \text{if student } i \text{ is left-handed (success)} \\ 0 & \text{if student } i \text{ is not left-handed (failure)} \end{cases}$ for $i = 1$ to 10.

Clearly, $P(Y_i = 1) = 0.15$ for $i = 1$ to 10.

Thus, Y_i for $i = 1$ to 10 are independent Bernoulli variables with parameter $p = 0.15$.

Hence, $X = \sum_{i=1}^{i=10} Y_i$ is the sum of 10 independent Bernoulli variables.

Therefore X must be a Binomial variable with parameters $n = 10$ and $p = 0.15$.

Probability mass function for X is $p(x) = \binom{10}{x} 0.15^x 0.85^{10-x}$ $x = 1, 2, 3, \ldots, 10$.

Expected value for X = $np = 1.5$ and variance for X = $npq = 1.275$.

Example 12.8

In a group of 100 students, it is known that 15 of these students are left-handed. Ten students are randomly chosen from this group. Define X: Number of left-handed students selected.
Show clearly that X cannot be a Binomial variable.

Solution:

X: Number of left-handed students out of the 10 selected.

Let $Y_i = \begin{cases} 1 & \text{if student } i \text{ is left-handed (success)} \\ 0 & \text{if student } i \text{ is not left-handed (failure)} \end{cases}$ for $i = 1$ to 10.

$P(Y_1 = 1) = 0.15$.

But $P(Y_2 = 1) = \frac{15}{99} \approx 0.15$ if the first student selected was not left-handed

or $P(Y_2 = 1) = \frac{14}{99} \approx 0.14$ if the first student selected was left handed.

Hence, Y_i for $i = 1$ to 10 are <u>*not*</u> independent Bernoulli variables.

Therefore, $X = \sum_{i=1}^{i=10} Y_i$ cannot be a Binomial variable.

Note:

- *If the 10 students were drawn from a group of 50 000 students, 3000 of whom are left handed, then the probabilities $P(Y_i = 1) \approx 0.15$ for $i = 1$ to 10. In this instance, Y_i for $i = 1$ to 10 would be approximately independent Bernoulli variables and $X = \sum_{i=1}^{i=10} Y_i$ could be approximated by a Binomial variable.*

Example 12.9

It is known that 5% of a batch of computer chips are defective. A sample of 20 chips is randomly selected from this batch.

(a) Find the probability that there are no more than 2 defective chips in this sample.

(b) Find the probability that there is at least one defective chip in this sample.

(c) Find the probability that there is no more than 2 defective chips in this sample if it is known that there is at least 1 defective chip in this sample.

(d) Find the expected number of defective chips in a sample of 1000 chips and its associated standard deviation.

Solution:

Let X: Number of defective chips in the sample of 20.

Let $Y_i = \begin{cases} 1 & \text{if chip } i \text{ is defective (success)} \\ 0 & \text{if chip } i \text{ is not defective (failure)} \end{cases}$ for $i = 1$ to 20.

Clearly, $P(Y_i = 1) = 0.05$ for $i = 1$ to 20.

Hence, $X = \sum_{i=1}^{i=20} Y_i$ is the sum of 20 independent Bernoulli variables with $p = 0.05$.

Therefore X must be a Binomial variable with parameters $n = 20$ and $p = 0.05$.

(a) $P(X \le 2) = 0.92452 \approx 0.9245$

```
binomialCDf(0,2,20,0.05)
                0.9245163262
binomialCDf(1,20,20,0.05)
                0.6415140776
binomialCDf(1,2,20,0.05)
                0.5660304038
binomialCDf(1,2,20,0.05)
binomialCDf(1,20,20,0.05)
                0.8823351249
```

(b) $P(X \ge 1) = (1 \le X \le 20)$
$= 0.64151 \approx 0.6415$

(c) $P(X \le 2 \mid X \ge 1) = \dfrac{P(1 \le X \le 2)}{P(X \ge 1)}$
$= \dfrac{P(1 \le X \le 2)}{P(1 \le X \le 20)}$
$= \dfrac{0.56603}{0.64151}$
$= 0.88234 \approx 0.8823$

(d) Let W: Number of defective chips in the sample of 1000.
Clearly $W \sim \text{Binomial}(n = 1000, p = 0.05)$.
Hence, $E(W) = 1000 \times 0.05 = 50$.
$Var(W) = 1000 \times 0.05 \times 0.95 = 47.5$
Standard deviation $= \sqrt{47.5} \approx 6.8920$.

Example 12.10

It is known that 65% of nurses in a state are members of the Nurses Union. Two samples each of size 12 nurses are selected. Find the probability that there are:
(a) exactly 8 nurses who are members of the Nurses Union in each sample.
(b) exactly 16 nurses who are members of the Nurses Union between the two samples.

Solution:

Let X: No. of nurses who are members of the Nurses Union in first sample (out of 12).

Let $N_i = \begin{cases} 1 & \text{if nurse } i \text{ is a union member (success)} \\ 0 & \text{if nurse } i \text{ is not a union member (failure)} \end{cases}$ for $i = 1$ to 12.

Clearly, $P(N_i = 1) = 0.65$ for $i = 1$ to 12.

Hence, $X = \sum_{i=1}^{i=12} N_i$ is the sum of 12 independent Bernoulli variables with $p = 0.65$.

Therefore: $X \sim B(12, 0.65)$.

Let Y: No. of nurses who are members of the Nurses Union in second sample (out of 12).

Similarly, $Y \sim B(12, 0.65)$.

(a) Required probability $= P(X = 8 \cap Y = 8)$
$= P(X = 8) \times P(Y = 8)$ *since X and Y are independent*
$= 0.23669 \times 0.23669 = 0.0560.$

(b) Required probability $= P(X + Y = 16)$

$$X + Y = \sum_{i=1}^{i=12} N_i + \sum_{i=1}^{i=12} M_i = \sum_{i=1}^{i=24} N_i$$

Hence $X + Y \sim B(n = 24, p = 0.65)$.
Therefore, $P(X + Y = 16) = 0.1682$.

Example 12.11

It is known that 4% of a batch of canned beef stew is contaminated with horse meat. A sample of n cans was randomly selected from this batch. Find the maximum value of n so that the probability that there is at least one contaminated can is no more than 50%.

Solution:

Let X: number of contaminated cans in the sample of n.
The probability of a can being contaminated is 0.04 which is the same for each can.
Hence $X \sim B(n, 0.04)$

It is required that $P(X \geq 1) \leq 0.5$
Rewriting $1 - P(X = 0) \leq 0.5$
$\Rightarrow P(X = 0) \geq 0.5$
Therefore $0.96^n \geq 0.5 \Rightarrow n \leq 16.98$

Therefore a sample of no more than 16 cans should be taken so that the probability that there is at least one contaminated can is less than 50%.

Exercise 12.2

1. It is known that 12% of students at a certain school attend school each morning without having taken any breakfast. A student is randomly chosen from this school. Define $X = 1$ if the student chosen attended school without breakfast and $X = 0$ otherwise. Find the expected value of X and its associated standard deviation.

2. It is estimated that 27% of 12 year olds in a certain state are classified as "over-weight". A 12 year old is randomly chosen from this state. Define $X = 0$ if the student chosen is not "overweight" and $X = 1$ otherwise. Find the expected value of X and its associated standard deviation.

3. It is estimated that 20% of students in a certain school ride their bicycles to school. Define $X_i = 1$ if student i rides to school and $X_i = 0$ otherwise, for $i = 1$ to 50. Let $Y = \sum_{i=1}^{50} X_i$. Describe the distribution for X_i and state its mean and variance. Describe the distribution for Y and state its mean and variance.

4. Assume that 95% of adults in a certain city own mobile phones. Let $X_i = 1$ if adult i owns a mobile phone and $X_i = 0$ otherwise, for $i = 1$ to 50. Let $Y = \sum_{i=1}^{100} X_i$. Describe the distribution for X_i and state its mean and variance. State the probability mass function for Y and state its mean and variance.

5. Assume that 65% of students at a certain school live within a 10 km radius of the school. Twenty students are randomly selected from this school. Define X: Number of students that live within a 10 km radius of the school. Show clearly that X is a binomial variable. State the probability mass function, expected value and variance for X.

6. A box contains 6 red balls and 14 white balls. Five balls were randomly selected from this box without replacement. Define X: Number of red balls selected. Show clearly that X cannot be a Binomial variable.

7. At a park and ride carpark, 78% of the 300 cars parked were Japanese branded cars. 20 cars were randomly selected. Define X: Number of Japanese branded cars selected. Show clearly that X cannot be a Binomial variable.

8. It is known that in a certain electoral district, 70% of the eligible voters are supporters of the Grey Party. A sample of 20 voters is chosen randomly from this electoral district. Let X: Number of Grey Party supporters in sample of 20.
   (a) Show that X must be a binomial variable. State the parameters of this variable.
   (b) Find the probability that the sample has: (i) at least 14 Grey supporters
      (ii) no more than 18 Grey supporters if it is known that there are at least 14 Grey supporters.
   (c) Find the most likely number of Grey supporters in this sample.

9. It is known that 30% of the teaching staff at a Primary School live at least 20 km from the school. A sample of 8 teachers is randomly chosen from this staff.
   Let X: Number of teachers that live at least 20 km from school in this sample.
   Show that X must be a binomial variable. State the parameters of this variable.
   Find the probability that within this sample there are:
   (a) exactly 5 teachers that live at least 20 km from the school
   (b) more than 2 but no more than 5 teachers that live at least 20 km from the school
   (c) not more than 4 teachers that live less than 20 km from the school
   (d) not more than 5 teachers that live at least 20 km from the school given that there are at least 3 of them.

10. It is known that 85% of the graduates of a certain University are placed in a job within six months. A sample of 15 graduates from this University is chosen at random.
    Let X: Number of graduates placed in a job within six months in the sample.
    Show that X must be a binomial variable. State the parameters of this variable.
    Find the probability that within this sample there are:
    (a) between 10 and 13 (inclusive) graduates that will be placed in a job within six months
    (b) at least 10 graduates that will be placed in a job within six months given that there are not more than 13 of them.

11. From previous records, it is known that there are 7 wet days in October. Find the probability that a randomly chosen week in October will have:
    (a) at most 3 wet days
    (b) at least 1 wet day given that there is at most 3 wet days.

12. It is known that 29% of the students in a primary school are asthmatic. Find the probability that in a random sample of 14 students from this school:
    (a) at least 3 students are asthmatic
    (b) at least 2 students are asthmatic given that less than 4 are asthmatic
    (c) the mean number of asthmatic students in this sample and its standard deviation.

13. James is the owner operator of Adventure West which runs one day treks to normally inaccessible parts of the bush. James' 4-wheel drive has seats for 10 paying customers. From experience, 30% of those who make reservations for the treks do not turn up. To offset the potential loss of earnings, James accepts up to 11 reservations for each trek. There were 11 bookings for the trek starting the 4th of March. Find the probability that:
    (a) there will be more customers than seats on for this trek
    (b) there will be more seats than customers for this trek.

14. 10% of DVD disks produced by a manufacturer are known to be defective.
    (a) Find the probability that in a random sample of 20 disks from the disks produced by this manufacturer 3 are defective.
    (b) Find the probability that in two random samples each of size 10 and 20 respectively, each sample has exactly 3 defective disks.
    (c) Find the most likely number of defective disks in a sample of 30 such disks.

15. Assume that 65% of households in a particular suburb have experienced a burglary within the last year. An insurance company randomly selects 2 samples each consisting of 20 households and 25 households respectively from this suburb.
Let X: No. of households that have been burgled within the last year in the sample of 20.
Let Y: No. of households that have been burgled within the last year in the sample of 25.
Show that X and Y must be binomial variables. State the parameters of these variables.
(a) Find the probability that each of these samples have exactly 13 households that have been burgled within the last year.
(b) Find the probability that between the two samples; exactly 26 households have experienced a burglary within the last year.

16. It is known that 85% of teachers in a particular state are members of the teachers' union. A random sample of 10 teachers from the state is chosen.
(a) Find the probability that there are exactly 7 teachers in this sample who are members of the teachers' union.
(b) Find the probability that a majority of the teachers in the sample are members of the teachers union.
(c) Find the probability that there are at least 5 teachers in the sample that are not members of the union given that there are at least 3 teachers in the sample who are members of the teachers' union.
(d) A second random sample of another 12 teachers is chosen. Find the probability that the each sample has 7 teachers who are members of the teachers' union.

17. It is known that without treatment 10% of patients diagnosed with a certain form of cancer will survive beyond 5 years. However, with treatment (which brings about severe side effects), 30% of patients with this form of cancer will survive beyond 5 years. A random sample (sample A) of 8 patients who were diagnosed at approximately the same time as suffering from this form of cancer and refusing any treatment is chosen. Another random sample (sample B) of 12 patients who were diagnosed at approximately the same time as those in sample A as suffering from this form of cancer and undergoing treatment is chosen. Find the probability that:
(a) at least 1 of the patients in sample A will survive beyond 5 years.
(b) exactly 3 of the patients in sample B will survive beyond 5 years given that no more than 10 die within 5 years.
(c) between the two samples; exactly 3 patients will survive beyond 5 years.

18. A chicken sexer has a 70% success rate in determining the sex of chickens. The chicken sexer is given a sample of 20 chickens.
(a) Find the probability that he successfully determines the sex of exactly 14 chickens.
(b) Find the probability that he successfully determines the sex of between 12 and 16 chickens.
(c) The chicken sexer is given 50 samples of 20 chickens each. Find the probability that he successfully determines the sex of between 12 and 16 chickens in exactly 30 samples.
(d) Find the mean number of chickens that are correctly sexed in a sample and the mean number of samples with between 12 and 16 chickens correctly sexed within the 50 samples in (c).

19. A golfer has a 75% chance of successfully putting a shot from a distance of 3 metres.
    (a) In a practice session consisting of 20 shots from a distance of 3 metres, find the probability that the golfer successfully putts at least 18 shots given that there were no more than 8 unsuccessful putts.
    (b) What is the minimum number of 3 metre putts the golfer must make so that there is a probability of more than 99% that he successfully putts at least one of them.
    (c) What is the maximum number of 3 metre putts the golfer must make so that the probability that he unsuccessfully putts at least one of them is no more than 99%.

20. A quiz is made up of 5 multiple choice questions. Each question is provided with 5 answers of which only one is correct. Richard randomly selects an answer to each of the 5 questions. To compensate for guessing, the teacher allocates 5 marks to each correctly answered question and deducts 1 mark for each incorrectly answered question. The minimum mark for the test is 0.
    (a) Determine the probability distribution for the random variable:
        (i) X that is defined as the number of correctly answered questions in this test
        (ii) T that is defined as the total number of marks scored by Richard.
    (b) A pass mark is set at 40% and over. Find the probability that Richard passes this test.
    (c) What is Richard's expected mark?

21. It is known that 10% of computer systems (type A) installed by Computer West encounter some form of malfunction within a week of installation. The average cost to Computer West of rectifying a malfunctioning system (type A) is $500. Built into the cost of installing each system is a sum of $400 for rectifying faults occurring within the first week. Six such systems were installed on a given work day.
    (a) Find the probability distribution for random variable:
        (i) M defined as the number of systems out of this six that will have a malfunction within one week of installation
        (ii) P defined as the profit for Computer West for rectifying malfunctioning systems.
    (c) Find the probability that Computer West suffers a lost in the money set aside for fault rectification within the first week from the six installed systems.
    (d) What is the expected profit for Computer West for the 6 installed systems?

13 Logarithms

13.1 Introducing Logarithms

- $3^{\square} = 81$

  ↑

  This number to which 3 must be raised, to equal 81, is called the logarithm of 81 to base 3

  - Clearly, this number is 4.
  - Hence, the logarithm of 81 to base 3 is 4.
  - We write $\log_3 81 = 4$.

- $2^{\square} = 32$

  ↑

  This number to which 2 must be raised, to equal 32, is called the logarithm of 32 to base 2

  - Clearly, this number is 5.
  - Hence, the logarithm of 32 to base 2 is 5.
  - We write $\log_2 32 = 5$.

- $5^{\square} = \frac{1}{25}$

  ↑

  This number to which 5 must be raised, to equal 1/25, is called the logarithm of 1/25 to base 5

  - Clearly, this number is -2.
  - Hence, the logarithm of $\frac{1}{25}$ to base 5 is -2.
  - We write $\log_5 \frac{1}{25} = -2$.

- $16 = 2^4$

  Hence, $\log_2 16 = 4$.

- $125 = 5^3$

  Hence, $\log_5 125 = 3$.

- In general, for real number $a > 0$, if,

  $M = a^p$ [exponential form]

  $\Rightarrow \log_a M = p$ [logarithmic form]

- Conversely, for real number $b > 0$, if,

  $\log_b N = q$ [logarithmic form]

  $\Rightarrow N = b^q$ [exponential form]

- Note that for real number $a > 0$, $a^0 = 1 \Rightarrow \log_a 1 = 0$.

  That is, the logarithm of the number *one* to any positive real number base is always *zero*.

- Also, for real number $a > 0$, $a^1 = a \Rightarrow \log_a a = 1$.

  That is, the logarithm of a given positive number with the same number as its base is always *one*.

- Also, for real number $a > 0$, $a^n = a^n \Rightarrow \log_a a^n = n$.

  That is, the logarithm of a given positive number to a certain power with the same number as its base is always the *power* to which the base is raised.

Example 13.1

Without using a calculator, evaluate: (a) $\log_3 27$ (b) $\log_{64} 8$ (c) $\log_{10} 0.01$

Solution:

(a) Let $\log_3 27 = x \Rightarrow 27 = 3^x$

Rewrite as $3^3 = 3^x \Rightarrow x = 3$

Therefore $\log_3 27 = 3$.

(b) Let $\log_{64} 8 = x \Rightarrow 8 = 64^x$

Rewrite as $8 = (8^2)^x \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}$

Therefore $\log_{64} 8 = \frac{1}{2}$.

(c) Let $\log_{10} 0.01 = x \Rightarrow 0.01 = 10^x$

Rewrite as $10^{-2} = 10^x \Rightarrow x = -2$

Therefore $\log_{10} 0.01 = -2$.

Exercise 13.1

1. Rewrite in exponential form:

   (a) $\log_3 9 = 2$ (b) $\log_{10} 1\,000 = 3$ (c) $\log_{25} 625 = 2$ (d) $\log_4 2 = \frac{1}{2}$

2. Rewrite in logarithmic form:

   (a) $10^4 = 10\,000$ (b) $2^6 = 64$ (c) $\frac{1}{4} = 2^{-2}$ (d) $9 = \sqrt{81}$

3. Without the use of a calculator, evaluate:

(a) $\log_4 16$ (b) $\log_2 8$ (c) $\log_3 243$ (d) $\log_7 343$

(e) $\log_{10} \frac{1}{10}$ (f) $\log_{11} \frac{1}{121}$ (g) $\log_4 0.25$ (h) $\log_{10} 0.000\,1$

4. Without the use of a calculator, solve:

(a) $\log_2(1 + x) = 2$ (b) $\log_3 (1 - x) = 4$ (c) $\log_5 \frac{1}{x+1} = -2$

(d) $\log_{10} (x^2 - 1) = 3$ (e) $\log x = \log (2x + 1)$ (f) $\log_2 x = \log_2 x^2$

13.1.1 Logarithms to base 10

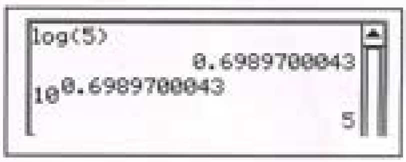
- Logarithms to base 10 are commonly referred to as common logarithms and are denoted **log** *without any subscript.*

Hands On Task 13.1

In this task, we will learn to use the **log** "button" on your calculator.

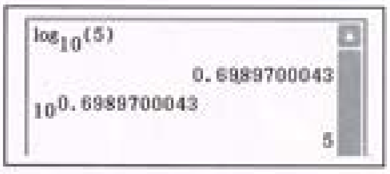
1. Use the **log** "button" on your calculator to evaluate:

x	5	600	0.8	0.07
$\log x$				



2. Use the $\mathbf{10^x}$ "button" which reverses the **log** "operation" to evaluate:

$\log x$	−3	−5.12	0.69897	5.1782
x				



Notes:

- *The **log x** and $\mathbf{10^x}$ functions are a pair of **inverse functions**. One function "undoes" what the other function does. Hence, to find a number whose log (to base 10) is 3.323, we use the $\mathbf{10^x}$ button.*

13.1.2 Change of Base

- CAS/graphic calculators have built in routines for evaluating logarithms with bases of 10, e and any other positive integer.

- Logarithms of one base are related to logarithms of another base through the following formula:

$$\log_a \mathrm{M} = \frac{\log_b \mathrm{M}}{\log_b a}$$

Example 13.2

Express $\log_5(15)$ exactly in terms of the common logarithm. Hence, evaluate $\log_5(15)$ correct to four decimal places.

Solution:

$$\log_5(15) = \frac{\log 15}{\log 5}$$
$$= 1.6826$$

Evaluated directly on a CAS calculator:

$\log_5(15)$ 1.6826

13.1.3 Graphs of Logarithmic Functions

Hands On Task 13.2

In this task, we will explore the main features of the graphs of logarithmic functions.

1. Use your CAS/graphic calculator to graph $y = \log x$. Use built in routines to find:
   (a) the intercepts, turning points and asymptotes (if any)
   (b) the value of x for which $y = 1$ and the values of x for which the curve does not exist.

2. Complete the following table.

x	1/4	1/2	1	2	4	8
$y = \log_2 x$						

   (a) Sketch the graph of $y = \log_2 x$.
   (b) Find the intercepts, turning points and asymptotes (if any).
   (c) Find the value of x for which $y = 1$.
   (d) Find the values of x for which the curve does not exist.

3. Complete the following table.

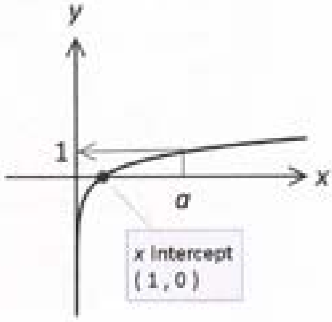
x	1/9	1/3	1	3	9	12
$y = \log_3 x$						

   (a) Sketch the graph of $y = \log_3 x$.
   (b) Find the intercepts, turning points and asymptotes (if any).
   (c) Find the value of x for which $y = 1$.
   (d) Find the values of x for which the curve does not exist.

4. Use your observations in Questions 1, 2 and 3 to describe the main features of the sketch of $y = \log_a x$.

5. Without using a table or a calculator, sketch $y = \log_5 x$. Indicate the intercepts, turning points and asymptotes (if any). Indicate also the point where $y = 1$.

△ Summary



- The curve $y = \log_a x$ (where $a > 0$) is:
  - defined only for $x > 0$.
  - has a vertical asymptote with equation $x = 0$.
  - has a horizontal intercept at $(1, 0)$ since when $x = 1$, $y = \log_a 1 = 0$.
  - passes through the point $(a, 1)$. since when $x = a$, $y = \log_a a = 1$.

    [That is, $y = 1$ when x = base of the Logarithm]

Exercise 13.2

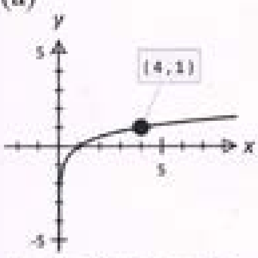
1. Without the use of a calculator, express each of the following in terms of log.

   (a) $\log_4(56)$ (b) $\log_3(50)$ (c) $\log_5(1/12)$

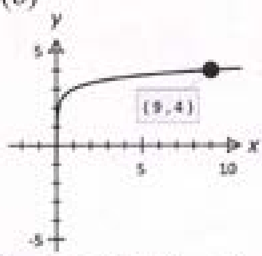
   (d) $\log_2(4.9)$ (e) $\log_7(3.4)$ (f) $\log_{11}(21/13)$

2. Match each of the following graphs which an equation from the given list.

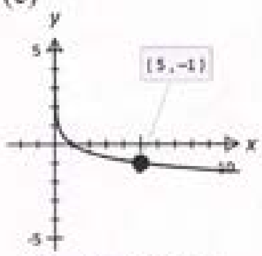
   (a)



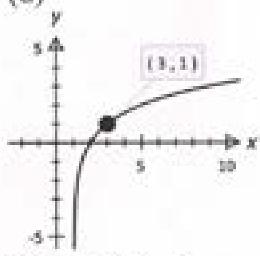
   (b)



   (c)



   (d)



   I $y = 3 + \log_9 x$ II $y = \log_2(x - 1)$ III $y = \log_3 x$ IV $y = \log_4 x$

   V $y = -\log_5 x$ VI $y = (\log_2 x) - 1$ VII $y = \log_2 x$ VIII $y = \log_{\frac{1}{5}} x$

3. Sketch each of the following curves. Indicate clearly, the asymptote(s), intercept(s) and one obvious point.

   (a) $y = \log_6 x$ (b) $y = \log_9(x + 4)$ (c) $y = -\log_7 x$

   (d) $y = \log_7(x - 1)$ (e) $y = (\log_2 x) - 2$ (f) $y = 3 - 2\log_3(x + 1)$

13.2 Rules for Logarithms

Hands On Task 13.3

In this task, we will explore the rules that govern the use of logarithms.

1. Find: (a) $\log_2 8$, $\log_2 32$ and $\log_2(8 \times 32)$ (b) $\log_3 9$, $\log_3 81$ and $\log_3(9 \times 81)$

   (c) $\log_5 \frac{1}{5}$, $\log_5 \frac{1}{25}$ and $\log_5\left(\frac{1}{5} \times \frac{1}{25}\right)$ (d) $\log 3$, $\log 2$ and $\log(3 \times 2)$

2. Summarise your observations in Question 1 by writing an expression relating $\log_a(M \times N)$, $\log_a M$ and $\log_a N$.

3. Find: (a) $\log_2 16$, $\log_2 64$ and $\log_2 \frac{64}{16}$ (b) $\log_3 27$, $\log_3 3$ and $\log_3 \frac{3}{27}$

   (c) $\log_5 125$, $\log_5 25$ and $\log_5 \frac{125}{25}$ (d) $\log 7$, $\log 4$ and $\log \frac{7}{4}$

4. Summarise your observations in Question 3 by writing an expression relating $\log_a \frac{M}{N}$, $\log_a M$ and $\log_a N$.

5. Find: (a) $\log_2 32$, $\log_2 32^5$ and $5 \times \log_2 32$ (b) $\log_3 9$, $\log_3 9^{-1}$ and $-1 \times \log_3 9$

   (c) $\log_5 \frac{1}{5}$, $\log_5\left(\frac{1}{5}\right)^{\frac{1}{2}}$ and $\frac{1}{2} \times \log_5 \frac{1}{5}$ (d) $\log 20$, $\log 20^3$ and $3 \times \log 20$

6. Summarise your observations in Question 5 by writing an expression relating $\log_a M^p$, $\log_a M$ and p.

△ Summary

- For $a > 0$, $M > 0$ and $N > 0$

• $\log_a(M \times N) = \log_a M + \log_a N$
• $\log_a \frac{M}{N} = \log_a M - \log_a N$
• $\log_a M^p = p \times \log_a M$

Example 13.3

Rewrite as a sum/difference of separate logarithmic terms/numbers, removing all powers.

(a) $\log 10x^2$ (b) $\log\left(\dfrac{10x^2}{1+y}\right)$

Solution:

(a) $\log 10x^2 = \log 10 + \log x^2$
$= 1 + 2\log x$

(b) $\log\left(\dfrac{10x^2}{1+y}\right) = \log 10x^2 - \log(1+y)$
$= 1 + 2\log x - \log(1+y)$

Example 13.4

Express as a single logarithmic term.

(a) $\log(1+x) + 2\log(2-x)$ (b) $\dfrac{1}{2}\log(1-x) - \log(1+x^2)$ (c) $\dfrac{\log_3 81}{\log_3 27}$

Solution:

(a) $\log(1+x) + 2\log(2-x) = \log(1+x) + \log(2-x)^2$
$= \log(1+x)(2-x)^2$

(b) $\dfrac{1}{2}\log(1-x) - \log(1+x^2) = \log\,(1-x)^{\frac{1}{2}} - \log(1+x^2)$

$= \log\dfrac{(1+x)^{\frac{1}{2}}}{1+x^2}.$

(c) $\dfrac{\log_3 81}{\log_3 27} = \dfrac{\log_3 3^4}{\log_3 3^3}$
$= \dfrac{4\log_3 3}{3\log_3 3} = \dfrac{4}{3}.$

Example 13.5

Without the use of a calculator, solve for x: (a) $\log_5(x+1) = 2$ (b) $2\,\log_3(x+1) = 3$

Solution:

(a) $\log_5(x+1) = 2$
$x + 1 = 5^2$
$x = 24$

(b) $2\,\log_3(x+1) = 3$
$\log_3(x+1)^2 = 3$
$(x+1)^2 = 3^3$
$x = -1 + 3\sqrt{3}$
(reject $-1 - 3\sqrt{3}$ as $x > -1$)

Exercise 13.3

1. Rewrite as a sum/difference of separate logarithmic terms/numbers, removing all powers:
   (a) $\log_5 5x$ (b) $\log_5 25x$ (c) $\log_5 (5/x)$ (d) $\log_5 1/(5x)$
   (e) $\log_3 x^2 y$ (f) $\log_3 (x/y^2)$ (g) $\log_2 x(1+x)^{1/2}$ (h) $\log_2 x/(1-x)^{1/2}$

2. Rewrite as a sum/difference of separate logarithmic terms/numbers, removing all powers:
   (a) $\log_5 x^5$ (b) $\log_5 5^x$ (c) $\log_3 9x$ (d) $\log_3 9^x$
   (e) $\log_7 \dfrac{1}{49^x}$ (f) $\log_7 49^{1+x}$ (g) $\log \dfrac{1}{100^{x+1}}$ (h) $\log \sqrt{10^x}$

3. Rewrite as a single logarithmic term:
   (a) $\log x + \log(1+x)$ (b) $\log x + \log y$ (c) $\log y + \log(1+x)$
   (d) $\log(1+x) + \log(1-x)$ (e) $\log x - \log(1+x)$ (f) $\log x - \log y$
   (g) $\log(1+x) - \log y$ (h) $\log(1-x^2) - \log(1+x)$

4. Rewrite as a single logarithmic term:
   (a) $\log x + 2\log x$ (b) $\log x + 2\log y$
   (c) $\log x - \dfrac{1}{2}\log(x+1)$ (d) $\dfrac{1}{2}\log y - \log(x+1)$
   (e) $\dfrac{1}{2}[\log(x+1) + \log(1-x^2)]$ (f) $\dfrac{1}{2}\log(x+1) - 2\log(1+x^2)$
   (g) $\dfrac{1}{2}\log x + \log(1+x) - 2\log(1-x)$ (h) $-\dfrac{1}{2}\log y + 2\log z - \dfrac{1}{3}\log x$

5. Simplify:
   (a) $\dfrac{\log_2 16}{\log_2 32}$ (b) $\dfrac{\log_5 125}{\log_5 \frac{1}{25}}$ (c) $\dfrac{\log_3 3^x}{\log_3 27}$ (d) $\dfrac{\log 8^{2x}}{\log 8^x}$

*6. Rewrite as a single logarithmic term:
   (a) $(\log_3 x) + 1$ (b) $(\log_5 x) - 1$ (c) $(\log_5 x) + 2$ (d) $(\log_9 x) - \dfrac{1}{2}$

7. Given that $\log_2 3 = p$ and $\log_2 5 = q$, find in terms of p and q:
   (a) $\log_2 15$ (b) $\log_2 75$ (c) $\log_2 0.6$
   *(d) $\log_2 6$ *(e) $\log_2 1.5$ *(f) $\log_2 120$

8. Given that $p = \log_3 15$ and $q = \log_3 5$, write as a single logarithmic term:
   (a) $p + q$ (b) $p + 2q$ (c) $\frac{1}{2}(p+q)$
   (d) $2p - q$ *(e) $p - q + 1$ *(f) $2 + p - q$

9. Without the use of a calculator, solve for x:
   (a) $\log_2(x-2) = 5$ (b) $\log(x^2 + 19) = 2$ (c) $2\log_5(x+1) = 3$
   (d) $\log_x 32 = 5$ (e) $\log_{3x} 81 = 2$ (f) $\log_x(x+2) = 2$

13.3 Using Logarithms to solve Exponential Equations

- In this section, we will use the common logarithm to solve for x in $a^x = b$, where b need not necessarily be an exact power of a.

Example 13.6

Without the use of a calculator, solve for x in terms of the common logarithm:

(a) $2^x = 10$ (b) $20(5^{2x}) = 40$ (c) $2^{x+1} = 3^x$ (d) $10(7^{2x}) = 50(7^{x-1})$

Solution:

(a) $2^x = 10$

Take log on both sides $\log 2^x = \log 10$

$$x \log 2 = 1$$

$$x = \frac{1}{\log 2}.$$

(b) $20(5^{2x}) = 40 \quad \Rightarrow \quad 5^{2x} = 2$

Take log on both sides $\log 5^{2x} = \log 2$

$$2x \log 5 = \log 2$$

$$x = \frac{\log 2}{2 \log 5}.$$

(c) $2^{x+1} = 3^x$

Take log on both sides $\log 2^{x+1} = \log 3^x$

$$(x + 1) \log 2 = x \log 3$$

$$x \log 3 - x \log 2 = \log 2$$

$$x(\log 3 - \log 2) = \log 2$$

$$x = \frac{\log 2}{\log 3 - \log 2}$$

(d) $10(7^{2x}) = 50(7^{x-1}) \quad \Rightarrow \quad \frac{7^{2x}}{7^{x-1}} = 5$

$$7^{x+1} = 5$$

Take log on both sides $(x + 1) \log 7 = \log 5$

$$x = -1 + \frac{\log 5}{\log 7}$$

Exercise 13.4

1. Without the use of a calculator, solve for x in terms of the common logarithm:

   (a) $2^x = 15$ (b) $3^x = 10$ (c) $5^x > 15$ (d) $\left(\frac{1}{2}\right)^x \le \frac{1}{5}$

   (e) $2^{x+1} = 6$ (f) $5^{x-1} = 10$ (g) $\frac{1}{3^{x+1}} = 7$ (h) $\sqrt{4^{x+1}} = 10$

2. Without the use of a calculator, solve for x in terms of the common logarithm:

   (a) $\log_2 6 = x$ (b) $\log_3 10 = x$ (c) $\log_8 5 > x$ (d) $\log_3 \frac{1}{2} \le x$

   (e) $\log_2 7 = x + 1$ (f) $\log_5 8 = 2x - 1$ (g) $\log_7 \frac{1}{3} = 1 - 2x$ (h) $2 \log_5 \frac{1}{2} = x - 1$

3. Find the value of each of the following to 4 decimal places:

   (a) $\log_6 10$ (b) $\log_{11} 5$ (c) $\log_2 17$ (d) $\log_3 \frac{1}{4}$

4. Without the use of a calculator, solve for the pronumeral in terms of the common logarithm:

   (a) $2^x = 3^{x-1}$ (b) $2^{x+1} = 5^x$ (c) $2^x = 64^{x+1}$ (d) $5^n = 10^{2-n}$

   (e) $4^x = 3(5^x)$ (f) $4(3^t) = 5(2^t)$ (g) $2^{x+2} = 3(7^{2-x})$ (h) $4(5^{t-1}) = 3(7^{1-2t})$

   (i) $100(2^{0.01x}) = 50(2^{0.05x})$ (j) $500(3^{-0.05t}) = 1\,000(3^{-0.01t})$

5. Without the use of a calculator, solve for x in terms of the common logarithm:

   (a) $(2^x + 3)(2^x - 4) = 0$ (b) $(2^x - 5)(2^x + 1) = 0$

   (c) $(3^x + 1)(3^x - 7) = 0$ (d) $(3^x - 5)(2^x + 5) = 0$

   (e) $x(2^x - 8) = 0$ (f) $x(5^x - 10) = 0$

   (g) $2^x(x - 3) = 0$ (h) $3^{-x}(x + 4) = 0$

6. Without the use of a calculator, solve for x in terms of the common logarithm:

   (a) $3^{2x} - 6(3^x) + 8 = 0$ (b) $5^{2x} - 3(5^x) - 10 = 0$

   (c) $2^{2x+1} + 2^x - 1 = 0$ (d) $3^{2x+1} + 5(3^x) - 2 = 0$

13.4 Applications using logarithmic models

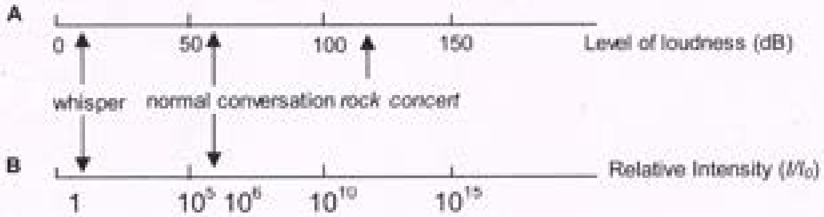
- Among the uses of logarithmic functions are the development of "*logarithmic scales*" that facilitate the comparison of intensities of natural phenomenon involving extremely large/small numbers.

Example 13.7

The intensity level or loudness level of sound is measured in decibels (dB). A sound with intensity $I_0 = 1 \times 10^{-12}$ Watts m^{-2} is arbitrarily denoted as 0 dB. This corresponds approximately with the faintest audible sound. The scale that defines the intensity levels (loudness) of various sounds compared to the faintest audible sound is given by:

$$\text{Intensity (loudness) level of sound with intensity } I = 10 \times \log\left(\frac{I}{I_0}\right).$$

Diagram A is a scale where the positions of various sounds are marked. Diagram B represents the relative intensities of the various sounds (compared to I_0).



A normal conversation is rated at 60 dB and is 10^6 times louder than the faintest audible sound.

(a) A whisper has an intensity 10 times I_0. What is the decibel rating for a whisper?

(b) Joe and Sue are standing just below the stage of a rock concert. The decibel rating of the music is 120 dB. How much louder is the music compared to a normal conversation?

(c) Comment on the linearity of the scales in diagrams A and B.

Solution:

(a) Decibel rating of whisper $= 10 \times \log\left(\frac{10 I_0}{I_0}\right) = 10 \times \log 10 = 10$

(b) Let I_m and I_c be the intensities for the rock concert and conversation respectively.

Hence $\quad 120 = 10 \times \log\left(\frac{I_m}{I_0}\right)$

$$\log\left(\frac{I_m}{I_0}\right) = 12 \quad \Rightarrow \quad \frac{I_m}{I_0} = 10^{12} \quad \Rightarrow \quad I_m = I_0 \times 10^{12}$$

Similarly $\quad I_c = I_0 \times 10^6$

Therefore $\quad \frac{I_m}{I_0} = \frac{I_0 \times 10^{12}}{I_0 \times 10^6} = 10^6$

Hence, the rock music is *one million times louder* than a normal conversation.

(c) The scale in Diagram A is linear whereas the scale in Diagram B is not linear.

Note:

- *In diagram B, the relative intensities ranged from 1 to 10^{12}. It is not possible to represent this on a linear scale. However, in diagram A, such a comparison is possible. The scale in diagram A is called a* ***logarithmic scale****. It permits the comparison of extremely large numbers using a linear scale.*

Exercise 13.5

1. The amount of light entering a camera is determined by the *f*-stop setting of the lens. The commonly used *f*-stop numbers are 1, 1.4, 2, 2.8, 4, 5.6, 8, 11, 16, 22, 32 and 45. Find the logarithm (base 10) of these *f*-stop numbers and comment on the distribution of these numbers.

2. To provide a schematic display of the various electromagnetic (EM) frequencies, a logarithmic scale is used. The scale is defined by: scale value = $\log_{10} f$, where f is the frequency of the EM radiation.
   (a) AM radio waves have frequencies between 5.4×10^5 and 1.6×10^6 Hertz. Find the scale value of AM radio waves on the logarithmic scale.
   (b) TV and FM radio waves have frequencies between 5.4×10^7 and 2.2×10^8 Hertz. Find the scale value of TV and FM radio waves.
   (c) X-rays have logarithmic scale values between 16.5 and 20.0. Consider the class of X-rays with a logarithmic scale value of 20.0 and visible light waves with a logarithmic scale value of 14.6. What is the ratio of the frequencies of X-rays to visible light in this case?

3. One of the scales used to measure the intensity of earthquakes is the Richter scale. The Richter scale is defined by the formula: $R = \log_{10} I_R$, where I_R is the relative intensity of the earthquake compared to what is taken as the smallest tremor that can be felt by humans. The San Francisco earthquake of 1906 is rated 8.25 on this scale.
   (a) The Meckering (Western Australia) earthquake of 1968 had a magnitude of 7.0. How much stronger is the San Francisco earthquake compared to the Meckering earthquake.
   (b) Find the Richter scale reading of an earthquake that is:
   (i) twice as strong, (ii) half as strong as the 1906 San Francisco earthquake.

4. The pH (pouvoir hydrogene - hydrogen power) of a solution is a measure of its hydrogen ion concentration. It is calculated using the formula: $pH = -\log_{10} H^+$, where H^+ is the concentration of hydrogen (H^+) ions in the solution (moles/litre). Pure water at 22° Celsius has a concentration of 1×10^{-7} moles/litre.
   (a) Calculate the pH of water at 22° Celsius.
   (b) Calculate the concentration of hydrogen ions in a solution with pH of 8.7.
   (c) Solution A has a pH of 9 whereas solution B has a pH of 3. Calculate the ratio of hydrogen ions in solution B to that in solution A.

5. The profit P (\$ thousand) in producing and selling x thousand items of a certain product is given by: $P = -0.8 + 10 \log (x + 1)$, for $x \geq 0$. Assume that all items produced are sold.
   (a) Find the number of items produced and sold when breakeven occurs.
   (b) Find the number of items produced and sold that correspond to a profit of \$20 000 .

6. The velocity, v (ms^{-1}), of a particle P moving in a straight line, t seconds after it passes a fixed point O is given by: $v = -1.2 \log(t + 1.5) + 1.6$, for $t \geq 0$.
   (a) Find the velocity of P when it first passes O.
   (b) Find when P first reverses its direction. *(c) Find when its speed is 1 ms^{-1}.

7. The displacement, s (m), of a particle P moving in a straight line, at time t seconds, from a fixed point O, is given by: $s = -2.3 + 1.5 \log(t + 2)$, for $t \geq 0$. Find:
   (a) the initial displacement of P (b) when the particle is at O
   (c) the displacement of P at $t = 40$ seconds
   *(d) the distance travelled in the first 40 seconds.

8. The unit cost, C ($), of producing a certain product is given by: $C = 2 + \log(x + 1)$, for $0 \leq x \leq 90\,000$, where x is the number of items manufactured. Each item is sold for $19.95. Assume that all items produced are sold. Find:
   (a) an expression for the cost of producing x items
   (b) an expression for the revenue when x items are sold
   (c) an expression for the profit when x items are produced and sold
   (d) the profit when 10 000 items are produced and sold
   (e) the number of items that need to be produced and sold to have a profit of $160 000 .

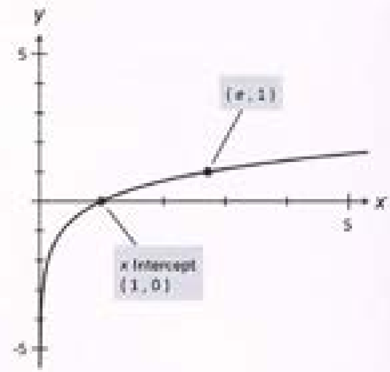
9. A small toy manufacturer produces 2 lines of toys; Cuddly Roos and Squeezy Bears. The profit, P ($ ten thousands) from the manufacture and sales of x thousand Cuddly Roos is given by $P_C = 2 \log(x + 0.75)$. The corresponding profit for Squeezy Bears is given by $P_S = 3 \log(x + 0.5)$. Assume that all toys produced are sold. Find:
   (a) the value of x which corresponds to the breakeven point for the production and sales of Cuddly Roos
   (b) the number of toys produced and sold for which the two lines of toys bring in the same profit
   *(c) the number of toys produced and sold for which Cuddly Roos will bring in more profit than Squeezy Bears.

10. A computer simulation is used to test the effectiveness of two different courses of treatment for a bacterial infection. For treatment course A, the number of bacteria, N (millions), t days after the start of the treatment, is given by: $N = 30.61 \log(15 - t)$, for $0 \leq t \leq 15$. For treatment course B, the corresponding model is given by: $N = 0.6(t + 5)(12 - t)$, for $0 \leq t \leq 12$.
    (a) Find the initial number of bacteria in both trials.
    (b) Find when the number of bacteria in the first treatment falls below 5 million.
    (c) Find when the two treatments have the same number of bacteria.
    (d) By comparing the graphs of the two models, discuss the differences between these two types of treatment and determine which is the more effective treatment.

14 Natural Logarithms

14.1 Natural Logarithms

- If $a = e^x$, where $a > 0$, then, $\log_e a = x$.
  $\log_e a$ is the logarithm of a to the base e.
- Logarithms with the base e are termed ***natural logarithms*** and are denoted *ln*.
  That is $\log_e a \equiv ln\, a$.
- The sketch of $y = ln\, x$ is given in the accompanying diagram. The obvious points are (1, 0) and $(e, 1) \approx (2.7, 1)$.



Hands On Task 14.1

In this task we will learn to use the **ln** button to calculate the natural logarithms of numbers.

1. Locate the **ln** button on your calculator.
2. Verify that $ln\, 2 = 0.693\ 147\ 181$ and $ln\, 0.5 = -0.693\ 147\ 181$.
3. Given that $ln\, 4 = 1.386\ 294\ 361$, find the calculator routine that will reverse the result.
4. Given that $ln\, 0.2 = -1.609\ 437\ 912$, find the calculator routine that will reverse the result.

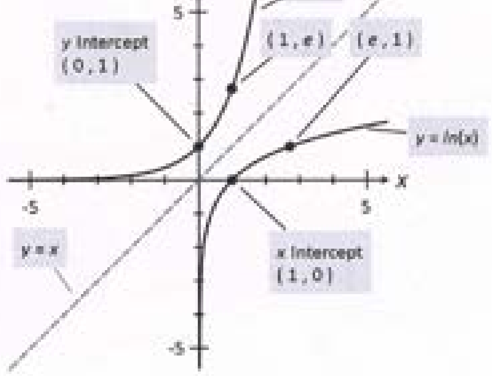
△ Summary

- In Hands on Task 13.1, we found that, $\log x$ reversed the effects of 10^x and vice-versa.
- Similarly, in this task, we found that, e^x reversed the effects of $ln\, x$.
- Therefore, e^x reverses the effects of $ln\, x$ and $ln\, x$ reverses the effects of e^x.
- Mathematically, a^x is described as the inverse of $\log_a x$, and
  $\log_a x$ is the inverse of a^x $(a > 0)$.

14.2 The Inverse Relationship between Logarithms and Exponentials

- Consider the real number a.
  - If we raise Euler's number e to the power of a we get e^a.
  - If we then take ln of the result, we get $ln\ e^a = a$.
  - In other words, we are back with the number we started with.
- Consider the positive real number a.
  - If we take ln of the number a, we get $ln\ a$.
  - If we raise Euler's number e to the result we get $e^{ln\ a}$.
    - Let $e^{ln\ a} = b$.
      Rewriting this in logarithmic form, $ln\ a = ln\ b$.
      $\Rightarrow \quad a = b$
      Hence, $e^{ln\ a} = a$.
  - In other words, we are back with the number we started with.
- In other words, the two functions $y = e^x$ and $y = ln\ x$ have an *inverse* relationship. One function "undoes" the effects of the other.
- Using this inverse relationship, we notice that:
  - $a = ln\ e^a$ for a as a real number
  - $a = e^{ln\ a}$ for a as a positive real number.

  The second result allows us to write any positive real number in terms of e.
- In general:
  - $a^x = ln\ e^{a^x}$ for a as a real number
  - $a^x = e^{x\ ln\ a}$ for a as a positive real number.
- The accompanying diagram shows the graphs of $y = e^x$ and $y = ln\ x$.
  - Notice that the graphs are symmetrical about the line $y = x$.
  - The symmetry is due to the inverse relationship where the x and y coordinates are swapped.
  - This feature is present in all pairs of functions that share an inverse relationship.



14.3 Solving expressions/equations with the use of natural logarithms

Example 14.1

Express $\log_5 13$ exactly in terms of the natural logarithm.

Solution:

Using the change of base formula:

$$\log_5 13 = \frac{\ln 13}{\ln 5}$$

Example 14.2

Express each of the following exactly as a power of Euler's number e:

(a) 10 (b) x^2

Solution:

(a) $10 = e^{\ln 10}$ (b) $x^2 = e^{2 \ln x}$

Example 14.3

Without the use of a calculator, solve for x in terms of the natural logarithm:

(a) $2^{-x} = \frac{1}{5}$ (b) $2^{x+1} = 3^{x-1}$

Solution:

(a)
$$2^{-x} = \frac{1}{5}$$
$$\ln 2^{-x} = \ln \frac{1}{5}$$
Hence
$$-x \ln 2 = -\ln 5$$
$$x = \frac{\ln 5}{\ln 2}$$

(b)
$$2^{x+1} = 3^{x-1}$$
$$\ln 2^{x+1} = \ln 3^{x-1}$$
$$(x+1)\ln 2 = (x-1)\ln 3$$
$$x(\ln 2 - \ln 3) = -(\ln 3 + \ln 2)$$
Hence:
$$x = \frac{\ln 6}{\ln \frac{3}{2}}$$

Example 14.4

Without the use of a calculator, solve for x:

(a) $e^{2x+1} = 4$ (b) $ln\,(x-1) = 3$ (c) $100\ e^{0.05x} = 500$

Solution:

(a)
$$e^{2x+1} = 4$$
$$\Rightarrow\ 2x+1 = ln\,4$$
$$x = \frac{ln\,4\ -\ 1}{2}$$
$$x = \frac{1}{2}\,ln\,4 - \frac{1}{2}$$
$$x = ln\,2 - \frac{1}{2}$$

(b)
$$ln\,(x-1) = 3$$
$$\Rightarrow\ x - 1 = e^3$$
$$x = 1 + e^3$$

(c)
$$100\ e^{0.05x} = 500$$
$$\Rightarrow\ e^{0.05x} = 5$$
$$0.05x = ln\,5$$
$$x = \frac{1}{0.05}\ ln\,5$$
$$x = 20\ ln\,5$$

Exercise 14.1

1. Simplify:
   (a) $e^{ln\,10}$ (b) $e^{2\,ln\,x}$ (c) $e^{-\,ln\,(1-x)}$
   (d) $e^{ln\,x^3}\ e^{\,ln\,(x-2)}$ (e) $ln\,(e^{x^2})$ (f) $ln\,(e^{x-1})$
   (g) $ln\,(e^x) - ln\,(e^{2x-1})$ (h) $ln\,(e^{-x}) + 2\ ln\,(e^x)$ (i) $2^{\log_2(5)}$
   (j) $10^{\log(7)}$ (k) $\log(10^{x+5})$ (l) $\log_3(3^{1-2x})$

2. Express each of the following in terms of the exponential number e:
   (a) 3 (b) 1/4 (c) 5^x (d) $\sqrt{x}$

3. Use natural logarithm to express each of the following as a power of 2:
   (a) 3 (b) 1/5 (c) x^2 (d) e^x

4. Express each of the following in terms of *ln*:

(a) $\log_4(56)$ (b) $\log_3(50)$ (c) $\log_5(1/12)$

(d) $\log_2(49)$ (e) $\log_7(3)$ (f) $\log_{11}(21/13)$

5. Without the use of a calculator, solve for x using natural logarithms:

(a) $e^x = 4$ (b) $e^x = \dfrac{1}{e^2}$ (c) $e^{-0.04x} = 5$

(d) $e^{1.2x} = 10$ (e) $10e^x = 5e^{1.1x}$ (f) $100e^{-0.05x} = 50e^{-0.01x}$

6. Without the use of a calculator, solve for x:

(a) $\ln x = 2$ (b) $\ln(1 - 2x) = 4$ (c) $2\ln x = 6$ (d) $\frac{1}{2}\ln(1 + x) = 2$

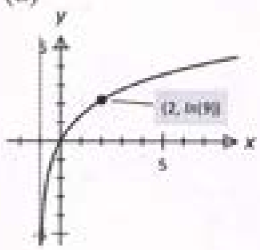
7. Without the use of a calculator, sketch each of the following. Indicate at least one obvious point. State the equation of the vertical asymptote.

(a) $y = \ln 2x$ (b) $y = \ln(x - 1)$ (c) $y = \ln(x + 2)$

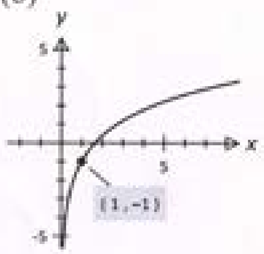
(d) $y = 1 + \ln x$ (e) $y = 1 - \ln x$ (f) $y = 1 + \ln(x - 1)$

8. Determine the equation of each of the following natural logarithm curves.

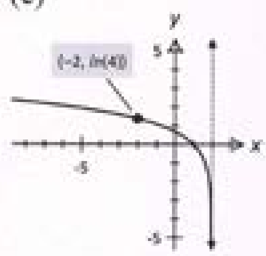
(a)



(b)



(c)



(d)

9. Without the use of a calculator solve for x in terms of *ln* where appropriate:

(a) $2^x = 1/200$ (b) $3^{-2x} = 30$ (c) $5^{x+1} = 1/80$

(d) $125^x = 10$ (e) $20(5^x) = 500$ (f) $100(3^{0.01x}) = 120$

(g) $20(4^x) = 30(2^x)$ (h) $50(5^x) = 40(3^{x-1})$ (i) $3^{x+1}\,8^{2x-1} = 5$

10. Without the use of a calculator solve for x in terms of *ln* where appropriate:

(a) $2(e^{2x}) - 5(e^x) + 2 = 0$ (b) $e^{2x} - 2e^x - 3 = 0$

(c) $4^x - 6(2^x) - 16 = 0$ (d) $2(9^x) + 3^{x-1} + 4 = 2(3^{x+1})$

14.4 Differentiating Logarithmic Functions

14.4.1 Differentiating *ln* (x)

Hands On Task 14.2

In this task, we will explore a rule for differentiating $ln\,(x)$ for $x > 0$.

Recall the definition of the derivative of $f(x)$ as $f'(x) = \lim_{h \to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$.

Consider $y = ln\,(x)$, $\Rightarrow \frac{d}{dx}ln\,(x) = \lim_{h \to 0}\left[\frac{ln\,(x+h)-ln\,(x)}{h}\right]$

1. Verify that $\lim_{h \to 0}\left[\frac{ln\,(x+h)-ln\,(x)}{h}\right]$ can be rewritten as $\lim_{h \to 0}\left[\frac{ln\,(1+\frac{h}{x})}{h}\right]$.

2. Rewrite $\frac{h}{x}$ as t. Verify that $\frac{d}{dx}ln\,(x) = \lim_{xt \to 0}\left[\frac{1}{x}\left(\frac{ln\,(1+t)}{t}\right)\right]$.

3. As x is a variable, restrict the limit $xt \to 0$ to $t \to 0$.
   Hence, $\frac{d}{dx}ln\,(x) = \lim_{t \to 0}\left[\frac{1}{x}\left(\frac{ln\,(1+t)}{t}\right)\right]$.
   As $\frac{1}{x}$ is not dependent on t, $\lim_{t \to 0}\left[\frac{1}{x}\left(\frac{ln\,(1+t)}{t}\right)\right] = \frac{1}{x} \times \lim_{t \to 0}\left[\frac{ln\,(1+t)}{t}\right]$
   Use your CAS calculator to find the value of $\lim_{t \to 0}\left[\frac{ln\,(1+t)}{t}\right]$ and hence find $\frac{d}{dx}ln\,(x)$.

△ Summary

Given that $y = ln\,(x)$, $\frac{dy}{dx} = \frac{1}{x}$ where $x > 0$.

14.4.2 Derivative of $\ln f(x)$

- Consider $y = \ln f(x)$.

  Let $u = f(x)$. Hence, $y = \ln(u)$.

  Using the chain rule:

$$\frac{dy}{dx} = \frac{1}{u} \times f'(x)$$

$$= \frac{f'(x)}{f(x)}$$

- Note the pattern:

$$y = \ln f(x)$$

$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

differentiate

Example 14.5

Find $\frac{dy}{dx}$ for: (a) $y = 2\ln(x)$ (b) $y = \ln(2x)$ (c) $y = \ln(x^2 + x)$

Solution:

(a)

$$y = 2\ln(x)$$

$$\Rightarrow \frac{dy}{dx} = 2\left(\frac{1}{x}\right) = \frac{2}{x}$$

(b)

$$y = \ln(2x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{2x} = \frac{1}{x}$$

(c)

$$y = \ln(x^2 + x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x+1}{x^2 + x}$$

Example 14.6

Find $\frac{dy}{dx}$ for: (a) $y = ln\,\sqrt{2x+1}$ (b) $y = ln\,[(x+1)^2(x+2)]$ (c) $y = ln\left[\frac{x-1}{x+1}\right]$

Solution:

(a)

$$y = ln\,\sqrt{2x+1}$$
$$= \frac{1}{2}\,ln\,(2x+1).$$
$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{2}{2x+1}\right) = \frac{1}{2x+1}$$

(b)

$$y = ln\,[(x+1)^2(x+2)]$$
$$= ln\,(x+1)^2 + ln\,(x+2)$$
$$= 2\,ln\,(x+1) + ln\,(x+2)$$
$$\Rightarrow \frac{dy}{dx} = \frac{2}{x+1} + \frac{1}{x+2}$$

(c)

$$y = ln\left[\frac{x-1}{x+1}\right]$$
$$= ln\,(x-1) - ln\,(x+1)$$
$$\Rightarrow \frac{dy}{dx} = \frac{1}{x-1} - \frac{1}{x+1}$$

Note:

- *It is often more convenient, using the rules of logarithm, to rewrite expressions before differentiating them.*

Example 14.7

Find $\frac{dy}{dx}$ for: (a) $y = x^2\,ln\,(x+1)$ (b) $y = \frac{ln\,(x)}{x}$

Solution:

(a)

$$y = x^2\,ln\,(x+1)$$
$$\Rightarrow \frac{dy}{dx} = 2x\,ln\,(x+1) + x^2\left(\frac{1}{x+1}\right)$$

(b)

$$y = \frac{ln\,(x)}{x}$$
$$\Rightarrow \frac{dy}{dx} = \frac{x\left(\frac{1}{x}\right) - ln\,(x)\times 1}{x^2} = \frac{1 - ln\,(x)}{x^2}$$

Exercise 14.2

1. Without using a calculator, differentiate with respect to x.

   (a) $y = ln\,(4x)$ (b) $y = ln\,(x/2)$ (c) $y = ln\,(x + 2)$ (d) $y = ln\,(2x + 1)$

   (e) $y = 4ln\,(1 + x^2)$ (f) $y = \frac{1}{2}\,ln\,(4 - x^2)$ (g) $y = ln\,(x/2 - 1)$ (h) $y = 4ln\,(x^3 + 1)$

2. Without using a calculator, differentiate with respect to x.

   (a) $y = ln\,(\sqrt{x})$ (b) $y = ln\,(1/x)$ (c) $y = ln\,(1 - x)^4$ (d) $y = ln\,(1 + 2x)^3$

   (e) $y = ln\,\sqrt{1+x}$ (f) $y = ln\left(\frac{1}{1+x}\right)$ (g) $y = ln\,\sqrt{1+x^2}$ (h) $y = ln\left[\frac{1}{(1+2x)^3}\right]$

3. Without using a calculator, differentiate with respect to x.

   (a) $y = ln\,[(x + 1)(x - 1)]$ (b) $y = ln\,[(x - 1)^3(x + 1)]$

   (c) $y = ln\,[(x^2 + 1)(x - 1)^4]$ (d) $y = ln\,[(x + 1)(x - 1)^{1/2}]$

   (e) $y = ln\left[\frac{x+2}{x-3}\right]$ (f) $y = ln\left[\frac{x+1}{x^2+1}\right]$

   (g) $y = ln\left[\frac{(2x+1)^2}{x-1}\right]$ (h) $y = ln\left[\frac{(x+1)^{1/2}}{x^3}\right]$

4. Without using a calculator, differentiate with respect to t.

   (a) $v = t^3\,ln\,(1 + t)$ (b) $v = 2t^2\,ln\,(1 - t)$ (c) $v = t\,ln\,(1 + t^2)$

   (d) $v = (1 + t)\,ln\,(1 - 2t)$ (e) $v = \frac{t}{ln(1-t)}$ (f) $v = \frac{1+t^2}{ln(t)}$

5. Find $f'(t)$.

   (a) $f(t) = \sqrt{t}\;ln\,(1 + 3t)$ (b) $f(t) = t\,ln\,(1 + t)^3$

   (c) $f(t) = t^2\,ln\,\sqrt{1-t}$ (d) $f(t) = \frac{ln(1+t)}{(t+1)}$

6. Find y'.

   (a) $y = ln\,(1 - e^x)$ (b) $y = ln\,[(1 - e^x)^2]$

   (c) $y = ln\,[(1 + e^x)(1 - e^{2x})]$ (d) $y = ln\left[\frac{1+e^x}{1-e^x}\right]$

   (e) $y = \frac{ln(x)}{e^x}$ (f) $y = (1 + e^x)\,ln\,(1 + x)$

7. Find y'.

   (a) $x^2\,ln\,[\sin(x)]$ (b) $\sin(x)\,ln\,[\sin(x)]$ (c) $\cos(t)\,ln\,[\sin(t)]$

   (d) $t\,ln\,[\tan(2t)]$ (e) $\frac{ln\,[\sin(t)]}{\sin(t)}$ (f) $e^{\sin x}ln\,[\cos(x)]$

8. Without the use of a calculator, find the equation of the tangent to the curve $y = x^2 \ ln\,(x+2)$ at the point where $x = -1$.

9. Without the use of a calculator, find the equation of the tangent to the curve $y = e^{2x}\ ln\,(x)$ at the point where $x = 1$.

10. Find the equation of the tangent to the curve $y = ln\,(x+1)$ at the point P where $x = 1$. Hence, find the equation of the line through P that is perpendicular to this tangent.

11. Find the coordinates of the point(s) on the curve $y = ln\,(x-2)$ where the gradient of the curve is 0.5.

12. The tangent to the curve $y = ln\,(1 + e^{-x})$ at the point P is parallel to the line $x + 2y = 10$. Find the coordinates of the point P.

13. Use calculus to find the coordinates of the stationary point on the curve with equation $y = 5\ ln\,(x) - x$. State the nature of this point.

14. A curve has equation $y = ln\,(1 + x^2)$.
    (a) Use calculus to find the coordinates of the:
        (i) stationary point and indicate the nature of this point (ii) inflection point.
    (b) Sketch this curve.

15. A curve has equation $y = 4x^2\ ln\,(0.5x)$ for $x > 0$.
    (a) Use calculus to find the coordinates of the:
        (i) stationary point and indicate the nature of this point (ii) inflection point.
    (b) Sketch this curve.

16. The annual profit, P hundred thousand dollars, of a retail store, is modelled by, $P = 2t\ ln\,(t)$, for $0 < t \le 10$, where t is time in years after establishing the store.
    (a) Find the instantaneous rate of change of profit with respect to time when $t = 1$.
    (b) Find when the rate of change of profit with respect to time is:
        (i) 0 (ii) \$400 000 per year.
    (c) Find the largest loss experienced by the store, and when it occurred.

17. A particle P moves along a straight line. Its velocity, v ms^{-1}, at time t seconds, is given by, $v = ln\,(t+1)$ for $t \ge 0$.
    (a) Find the initial velocity and acceleration of P.
    (b) Find the acceleration of P when its velocity is 2 ms^{-1}.
    (c) Find the average acceleration of P for the interval $0 \le t \le 10$.
    (d) Describe the acceleration of P for large values of t.

18. The displacement, s (m), of a particle P moving in a straight line, at time t seconds, from a fixed point O, is given by: $s = -2 + 1.5\, ln\,(t + 3)$, for $t \geq 0$. Find:
   (a) the initial displacement of P
   (b) when the particle is at O
   (c) the displacement of P at $t = 40$ seconds
   (d) the distance travelled in the first 40 seconds.

14.5 Anti-derivative of $\dfrac{f'(x)}{f(x)}$

- Since $\dfrac{d}{dx} ln\, f(x) = \dfrac{f'(x)}{f(x)}$,

$$\int \frac{f'(x)}{f(x)}\, dx = ln\left|f(x)\right| + C.$$

- In particular:

$$\int \frac{1}{x}\, dx = ln\left|x\right| + C.$$

Example 14.8

Find: (a) $\displaystyle\int \frac{1}{1+5x}\, dx$ (b) $\displaystyle\int \frac{4x^2}{2-3x^3}\, dx$

Solution:

(a)

$$\int \frac{1}{1+5x}\, dx = \frac{1}{5}\int \frac{5}{1+5x}\, dx$$
$$= \frac{1}{5} ln\left|1+5x\right| + C.$$

(b)

$$\int \frac{4x^2}{2-3x^3}\, dx = \frac{4}{-9}\int \frac{-9x^2}{2-3x^3}\, dx$$
$$= -\frac{4}{9} ln\left|2-3x^3\right| + C.$$

n a

Example 14.9

Find: (a) $\int \frac{e^{2x}}{1-e^{2x}}\,dx$ (b) $\int \tan(2x)\,dx$

Solution:

(a)
$$\int \frac{e^{2x}}{1-e^{2x}}\,dx = \frac{1}{-2}\int \frac{-2e^{2x}}{1-e^{2x}}\,dx$$
$$= -\frac{1}{2}ln\left|1-e^{2x}\right| + C.$$

(b)
$$\int \tan(2x)\,dx = \int \frac{\sin(2x)}{\cos(2x)}\,dx$$
$$= \frac{1}{-2}\int \frac{-2\sin(2x)}{\cos(2x)}\,dx$$
$$= -\frac{1}{2}ln\left|\cos(2x)\right| + C$$

Exercise 14.3

1. Integrate each of the following with respect to x.

(a) $\frac{2}{x}$ (b) $\frac{3}{4x}$ (c) $\frac{x^2+x}{x^2}$ (d) $\frac{1}{3+2x}$

(e) $\frac{(x+1)^2}{x}$ (f) $\left(1+\frac{1}{x}\right)^2$ (g) $\frac{2x}{1-5x^2}$ (h) $\frac{-4x^2}{2x^3-5}$

2. Find the anti-derivative of:

(a) $\frac{x+3}{x^2+6x}$ (b) $\frac{3x^2+x-1}{2x^3+x^2-2x}$ (c) $\frac{1}{2(1+\sqrt{x})\sqrt{x}}$

(d) $\left(x+\frac{1}{x^2}\right)^2$ (e) $\int \frac{-e^x+e^{-x}}{e^x+e^{-x}}\,dx$ (b) $\int \frac{5e^{-2x}}{1+e^{-2x}}\,dx$

3. Integrate each of the following with respect to the appropriate variable:

(a) $\tan(t)$ (b) $\tan(1+2x)$ (c) $\tan(1-x)$ (d) $\frac{1}{\tan(3x)}$

4. Integrate each of the following with respect to the appropriate variable:

(a) $\frac{\cos(\pi x)}{1+\sin(\pi x)}$ (b) $\frac{\sin(\pi x)}{2-\cos(\pi x)}$ (c) $\frac{1+\cos(2x)}{2x+\sin(2x)}$ (d) $\frac{\sin(t)-\cos(t)}{\sin(t)+\cos(t)}$

5. Verify that $\dfrac{1}{(x+1)(x+2)} = \dfrac{1}{x+1} - \dfrac{1}{x+2}$. Hence, find $\int \dfrac{1}{(x+1)(x+2)}\,dx$.

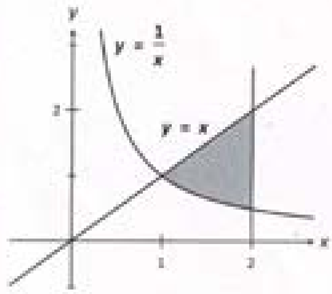
6. Given that $\dfrac{x+2}{(x+1)(x-2)} = \dfrac{A}{x+1} + \dfrac{B}{x-2}$ where A and B are constants, determine the values of A and B. Hence find $\int \dfrac{x+2}{(x+1)(x-2)}\,dx$.

7. The gradient function of a curve is given by $\dfrac{dy}{dx} = 4x + \dfrac{1}{1-2x}$. Find the equation of this curve given that it passes through the point (0, 4).

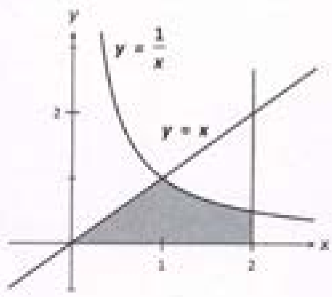
8. The gradient function of a curve is given by $\dfrac{dy}{dx} = e^x + \dfrac{x}{1+x^2} - 1$. Find the equation of this curve given that it passes through the point (0, −2).

9. Use calculus to find the area of the shaded region.

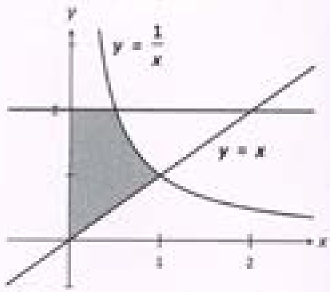
(a)



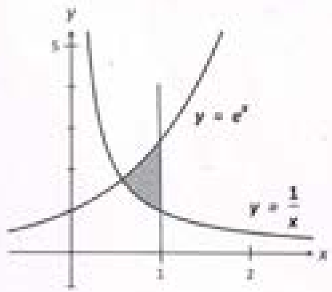
(b)



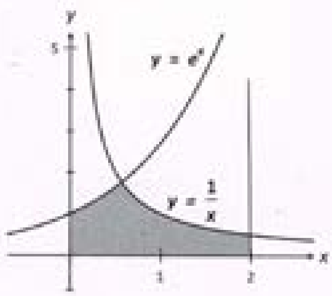
(c)



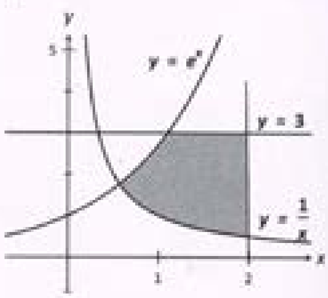
(d)



(e)



(f)



10. Use calculus to find the area trapped between the curve $y = \dfrac{x}{1+x^2}$, the x-axis and the lines $x = -2$ and $x = 2$.

15 Continuous Random Variables

15.1 Definitions

- A random variable with a range space that is continuous is called a *Continuous Random Variable* (CRV).
- For example, X: the height of year 12 students at Fremantle College.
  X takes values over an interval rather than discrete values.
  The values taken by X are usually obtained through the process of *"measurement"* rather than through the process of *"counting"*.

15.2 Estimating Probabilities

Example 15.1

The accompanying table shows the distribution of heights of a group of students. Define random variable X: The height of students. Use this table to estimate:

(a) $P(X < 170)$ (b) $P(X \geq 150 \mid X < 170)$
(c) $P(150 \leq X < 152)$ (d) $P(X < 152)$
(e) the expected value of X.

Height x cm	No. of Students
$140 \leq x < 145$	4
$145 \leq x < 150$	12
$150 \leq x < 155$	20
$155 \leq x < 160$	22
$160 \leq x < 165$	25
$165 \leq x < 170$	20
$170 \leq x < 175$	8
$175 \leq x < 180$	5
$180 \leq x < 185$	4

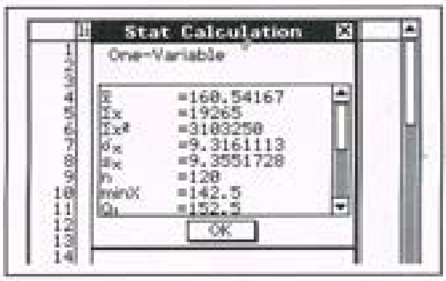
Solution:

(a) $$P(X < 170) = \frac{4+12+20+22+25+20}{4+12+20+22+25+20+8+5+4} = \frac{103}{120}$$

(b) $$P(X \geq 150 \mid X < 170) = \frac{20+22+25+20}{103} = \frac{87}{103}.$$

(c) Using linear interpolation:

$$P(150 \leq X \leq 152) = \frac{\left(\frac{2}{5}\times 20\right)}{120} = \frac{8}{120} = \frac{1}{15}$$



(d) $$P(X < 152) = P(X < 150) + P(150 \leq X \leq 152) = \frac{4+12}{120} + \frac{8}{120} = \frac{1}{5}$$

(e) Using class-midpoints as representative of class intervals, $E(X) = 160.54$.

Notes:

- *In parts (c) and (d), the method of* <u>*linear interpolation*</u> *is used to estimate the interval frequency. In this method, the number of students in a given class is assumed to be distributed uniformly within the class.*
- *The expected value of X is the same as the statistical mean for x. In estimating the statistical mean, all students in each class are assumed to be located at the mid-point of the respective class interval.*

Example 15.2

The accompanying table shows the distribution of masses of a group of males. Define random variable X: The mass of males.

(a) Complete the column of relative frequencies.
(b) Use this table to estimate k if $P(X < k) = 0.6$.

Mass x kg	No. of Males	Relative Frequencies
$40 \le x < 50$	10	
$50 \le x < 60$	15	
$60 \le x < 70$	18	
$70 \le x < 80$	28	
$80 \le x < 90$	16	
$90 \le x < 100$	13	

Solution:

(a) Total frequency = 100.
For $40 \le x < 50$:
relative frequency $= \dfrac{10}{100} = 0.1$.
Completed column is as displayed.

Mass x kg	No. of Males	Relative Frequencies
$40 \le x < 50$	10	0.1
$50 \le x < 60$	15	0.15
$60 \le x < 70$	18	0.18
$70 \le x < 80$	28	0.28
$80 \le x < 90$	16	0.16
$90 \le x < 100$	13	0.13

(b) $P(X < k) = 0.6$.
From table, $P(X < 70) = 0.43$
and $P(X < 80) = 0.71$.
Hence, $70 \le k < 80$.
$P(X < 70) + P(70 \le X < k) = 0.6$
$P(70 \le X < k) = 0.6 - 0.43 = 0.17$
Hence, number of students with height $70 \le X < k$ cm is $0.17 \times 100 = 17$.
Using linear interpolation:

$$\frac{k-70}{10} \times 28 = 17 \quad \Rightarrow \quad k \approx 76.1$$

Exercise 15.1

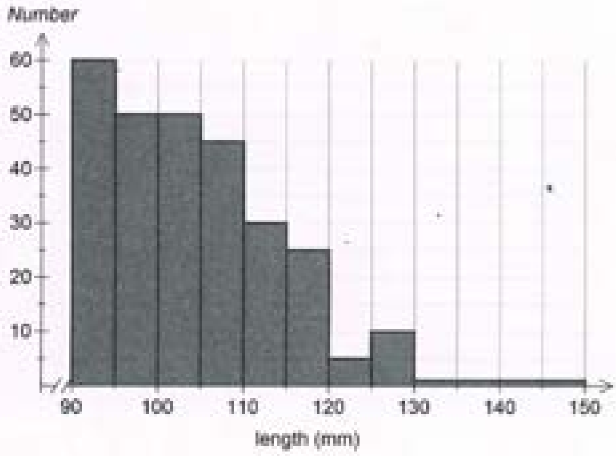
1. The accompanying table shows the distribution of heights of a group of students. Define random variable X: The height of students. Use this table to estimate:
   (a) $P(X \ge 120)$ (b) $P(X < 140 \mid X \ge 120)$
   (c) $P(X < 146)$ (d) k if $P(X < k) = 0.7$
   (e) the expected value and variance of X.

Height x cm	No. of Students
$100 \le x < 110$	22
$110 \le x < 120$	18
$120 \le x < 130$	32
$130 \le x < 140$	41
$140 \le x < 150$	19
$150 \le x < 160$	12
$160 \le x < 170$	6

2. The accompanying table shows the distribution of masses of a group of females. Define random variable X: The mass of females.
   (a) Complete the column of relative frequencies.
   (b) Use this table to estimate k if $P(X < k) = 0.25$.
   (c) Find the mass exceeded by 20% of the females.

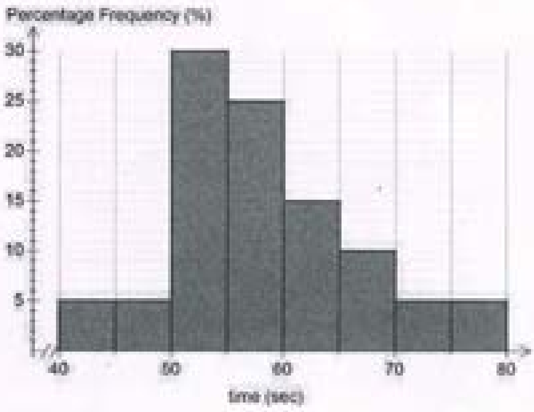
Mass x kg	No. of Students	Relative Frequencies
$40 \le x < 50$	50	
$50 \le x < 60$	68	
$60 \le x < 70$	27	
$70 \le x < 80$	10	
$80 \le x < 90$	5	

3. The accompanying histogram shows the distribution of the length of 279 blue swimmer crabs measured across the widest part of the shell (carapace length). The data was collected during the later part of the closed season (when no fishing of these crabs is permitted). During the open season only crabs with carapace lengths exceeding 127 mm are permitted to be caught. Define the random variable X: Carapace length of crab.



   (a) Estimate $P(X < 127)$.
   (b) Estimate k if $P(100 \le X \le k) = 0.5$.
   (c) Suggest with reasons if illegal fishing of these crabs have been occurring.

4. The accompanying histogram shows the distribution of times taken by a group of 100 students to complete a required task. Define the random variable T: Time to complete task. Estimate:



   (a) the expected value μ and the standard deviation σ for T.
   (b) $P(\mu - \sigma \le T \le \mu + \sigma)$.
   (c) the time exceeded by 75% of the students.

5. The lifetimes (t hours) of a consignment of electric light bulbs are displayed below.

t	$t \le 50$	$t \le 100$	$t \le 150$	$t \le 200$	$t \le 250$	$t \le 300$	$t \le 350$	$t \le 400$
Number of bulbs	8	20	60	180	250	300	380	410

   Define the random variable T: Lifetime of light bulbs. Estimate:
   (a) the mean and variance of T
   (b) the median m for T, that is $P(T \le m) = 0.5$
   (c) the lifetime exceeded by 10% of the light bulbs.

6. Air samples were taken at a research site for 150 consecutive days and the concentration of pollutants (c parts/million) recorded. The table below shows the results of the study.

c	$c \le 1$	$c \le 2$	$c \le 3$	$c \le 4$	$c \le 5$	$c \le 6$	$c \le 7$	$c \le 8$	$c \le 9$	$c \le 10$
Number of days	7	17	29	43	56	74	90	110	135	150

   Define the variable C: concentration of pollutants. Estimate:
   (a) $P(C \le 4.4)$
   (b) the median m for C, that is $P(C \le m) = 0.5$
   (c) the concentration of pollutants exceeded by 5% of the days in the study

15.3 The Probability Density Function

- Let X be a continuous random variable.
- The function $f(x)$ is defined as the *probability density function* for X where $a \le x \le b$ if:
  - $f(x) \ge 0 \qquad a \le x \le b$
  - $\int_a^b f(x)\,dx = 1$
- If $f(x)$ is the probability density function for X then:

  $$P(m \le X \le n) = \int_m^n f(x)\,dx \qquad \text{where } a \le m \le n \le b$$

- *Interpreted geometrically*, $f(x)$ is a probability density function for X where $a \le x \le b$, if:
  - the graph of $y = f(x)$ is always *on or above the x-axis* for $a \le x \le b$
  - the *area* between the curve $y = f(x)$ and the x-axis and between the vertical lines $x = a$ and $x = b$ is exactly *one*.
  - As a consequence:
    $P(m \le X \le n)$ = Area between the curve $y = f(x)$ and the x-axis and the vertical lines $x = m$ and $x = n$
- For continuous random variables, probabilities are defined by the use of integrals. As such, for any continuous random variable X, the probability that the X takes a specific value within its range space (e.g. X = 5) is zero as the integral over a point is zero. For continuous random variables only probabilities associated with intervals of values are meaningful.
- Note that as $P(X = k) = \int_k^k f(x)\,dx = 0$,

  $$P(X \le k) = P(X < k) + P(X = k) = P(X < k).$$

  That is, for continuous random variables, the "equal sign" has no impact on probability values.
- The mean of a continuous random variable with probability density function $f(x)$ with domain $a \le x \le b$ is given by $\mu = E(X) = \int_a^b x \times f(x)\,dx$.
- The variance of a continuous random variable with probability density function $f(x)$ with domain $a \le x \le b$ is given by $\sigma^2 = \text{Var}(X) = E(X^2) - [E(X)]^2$ where $E(X^2) = \int_a^b x^2 \times f(x)\,dx$.

Example 15.3

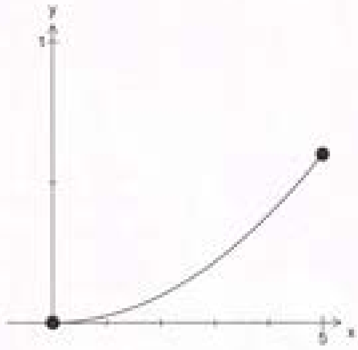
X is a continuous random variable with probability density function given by:

$f(x) = \dfrac{3x^2}{125}$ for $0 \le x \le 5$.

(a) Sketch $f(x)$.
(b) Verify that $f(x)$ is a probability density function.
(c) Find $P(1 \le X \le 3)$.
(d) Find $P(X \le 3)$
(e) Find $P(X \ge 1 \mid X \le 3)$
(f) Find the mean and variance of X.

Solution:

(a) The sketch of $f(x)$ is shown in the accompanying diagram.

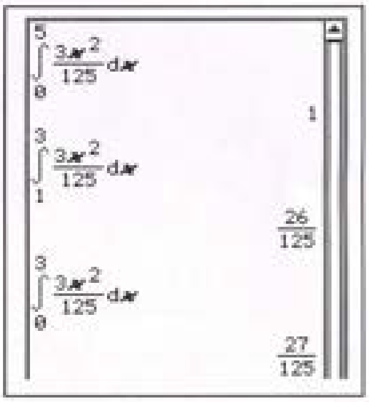


(b) Clearly, $f(x) = \dfrac{3x^2}{125} \ge 0$ for $0 \le x \le 5$.

[See graph for visual representation].

Also, $\displaystyle\int_0^5 \frac{3x^2}{125}\,dx = 1$.

Hence, $f(x) = \dfrac{3x^2}{125}$ where $0 \le x \le 5$ is a probability density function.



(c) $P(1 \le X \le 3) = \displaystyle\int_1^3 \frac{3x^2}{125}\,dx = \frac{26}{125}$.

(d) $P(X \le 3) = P(0 \le X \le 3) = \displaystyle\int_0^3 \frac{3x^2}{125}\,dx = \frac{27}{125}$.

(e) Using the conditional rule for probability:

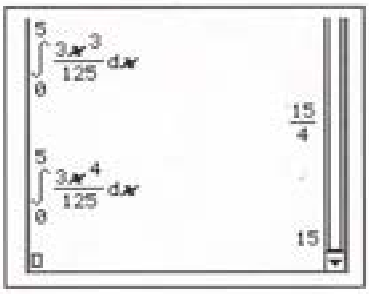
$$P(X \ge 1 \mid X \le 3) = \frac{P(X \ge 1 \cap X \le 3)}{P(X \le 3)} = \frac{P(1 \le X \le 3)}{P(X \le 3)}$$

$$= \frac{\left(\frac{26}{125}\right)}{\left(\frac{27}{125}\right)} = \frac{26}{27}.$$

(f) Mean value for X, $\mu = \displaystyle\int_0^5 \frac{3x^3}{125}\,dx = \frac{15}{4}$.

Variance for X, $\sigma^2 = \displaystyle\int_0^5 \frac{3x^4}{125}\,dx - \left(\frac{15}{4}\right)^2$

$= \dfrac{15}{16}$.



Example 15.4

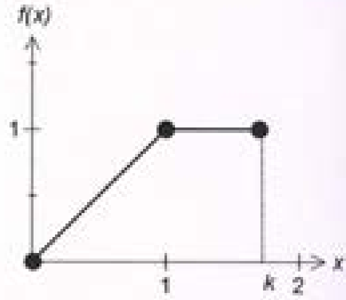
The probability density function $f(x)$ of a continuous random variable X defined for $0 \le x \le k$ is sketched below.

(a) Find the value of k.

(b) Find the probability density function of X.

(c) Find $P(X \le 1)$.

(d) Find m such that $P(X \le m) = 0.75$.



Solution:

(a) Using a geometrical approach, since $f(x)$ is a probability density function, the area of trapezium OABD = 1.
Hence using the trapezium rule:

$$\frac{1}{2} \times [k + (k-1)] \times 1 = 1 \qquad \Rightarrow \quad k = 1.5$$

(b) *For $0 \le x \le 1$, $f(x)$ is represented graphically by a straight line passing through the points (0, 0) and (1, 1). Equation of this line is $y = x$.*
For $1 < x \le 1.5$, $f(x)$ is represented graphically by the line $y = 1$.

Hence, the probability density function is given by:

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 1 & 1 < x \le 1.5 \end{cases}$$

(c) $P(X \le 1)$ = Area of triangle OAC

$= \frac{1}{2} \times 1 \times 1$

$= 0.5$

(d) Since $P(X \le 1) = 0.5$, m must necessarily be greater than 1.

Rewrite $\quad P(X \le m) = 0.5 + P(1 \le x \le m)$

Hence $\quad 0.5 + P(1 \le x \le m) = 0.75$

$\quad P(1 \le x \le m) = 0.25$

$P(1 \le x \le m)$ = Area of rectangle with length 1 and width $(m - 1)$

$= 1 \times (m-1) = m - 1$

Hence $\quad m - 1 = 0.25$

$\Rightarrow \quad m = 1.25.$

Example 15.5

The probability density function $f(x)$ of a continuous random variable X is given by

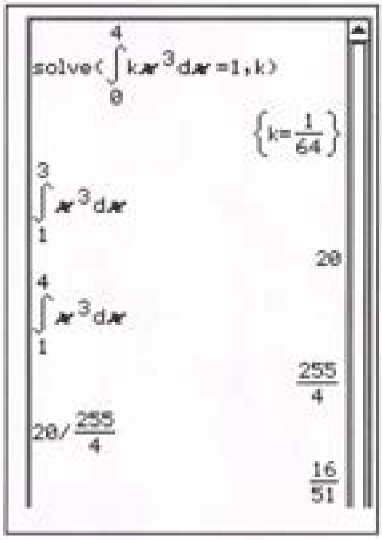
$$f(x) = \begin{cases} kx^3 & 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Find the value of k. (b) Find $P(X \le 3 \mid X \ge 1)$. (c) Find m such that $P(X \le m) = 0.75$.

Solution:

(a) Since $f(x)$ is a probability density function, $\int_0^4 kx^3\,dx = 1$.

Hence, $k\left[\dfrac{x^4}{4}\right]_0^4 = 1 \Rightarrow k = \dfrac{1}{64}$



(b) $P(X \le 3 \mid X \ge 1) = \dfrac{P(1 \le X \le 3)}{P(X \ge 1)}$

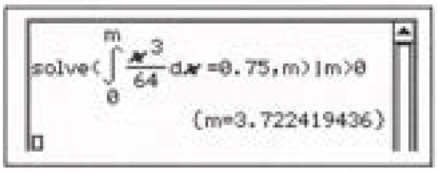
$$= \frac{k\int_1^3 x^3\,dx}{k\int_1^4 x^3\,dx} = \frac{20}{\left(\frac{255}{4}\right)} = \frac{16}{51}$$

(c) $P(X \le m) = 0.75 \Rightarrow \dfrac{1}{64}\displaystyle\int_0^m x^3\,dx = 0.75$

$$\frac{1}{64}\left[\frac{x^4}{4}\right]_0^m = 0.75$$

$$\frac{m^4}{256} = 0.75$$

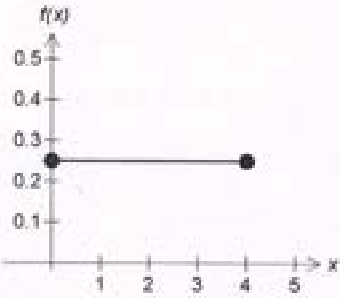
$$m \approx 3.72$$



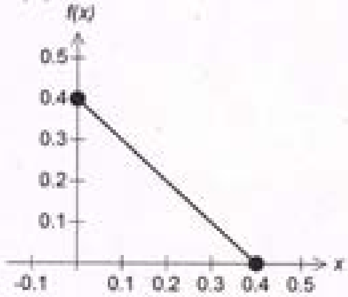
Exercise 15.2

1. Determine with reasons whether $f(x)$ with the corresponding graphs are probability density functions:

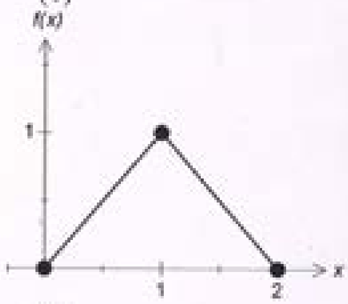
(a)



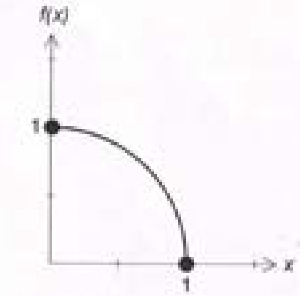
(b)



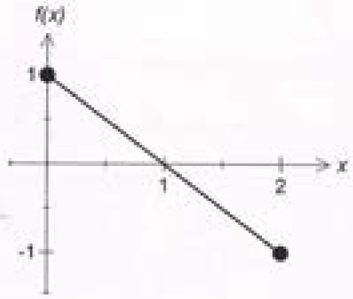
(c)



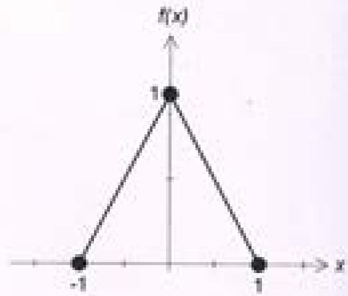
(d)



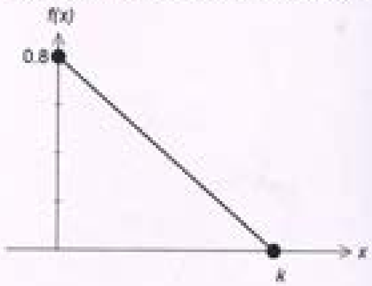
(e)



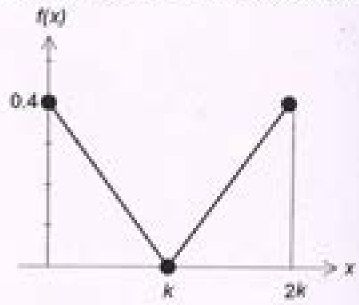
(f)



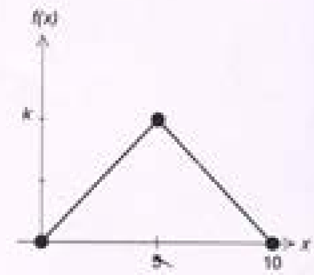
2. The probability density function of a random variable X defined for $0 \le x \le k$ is sketched below. Find:
   (a) the value of k
   (b) the probability density function of X
   (c) $P(X > 0.5k)$
   (d) $P(X > 0.75k)$
   (e) $P(X > 0.75k \mid X > 0.5k)$



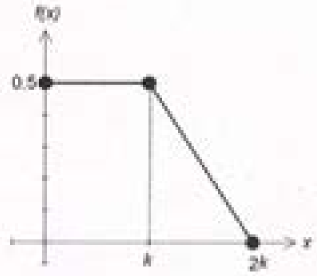
3. The probability density function of a random variable X defined for $0 \le x \le 2k$ is sketched below. Find:
   (a) the value of k
   (b) the probability density function of X
   (c) $P(X > 0.5k)$
   (d) $P(X < 0.75k)$
   (e) $P(X > 0.5k \mid X < 0.75k)$



4. The probability density function of a random variable X defined for $0 \le x \le 10$ is sketched in the accompanying diagram. Find:
   (a) the value of k
   (b) the probability density function of X
   (c) m such that $P(5 < X \le m) = 0.25$



5. The probability density function of a random variable X defined for $0 \le x \le 2k$ is sketched in the accompanying diagram. Find:
   (a) the value of k
   (b) the probability density function of X
   (c) m given that $P(k < X < m) = 0.2$.



6. Verify that $f(x) = 2x$ where $0 \le x < 1$ is a probability density function of a continuous random variable X. Hence find:
   (a) $P(X > 0.5)$
   (b) $P(X > 0.1)$
   (c) $P(X > 0.5 \mid X > 0.1)$
   (d) the mean and variance for X .

7. A continuous random variable X has probability distribution function given by $f(x) = kx^2$ where $0 \le x \le 2$. Find:
   (a) the value of k
   (b) $P(X \le x)$ where $0 \le x \le 2$
   (c) the value of m such that $P(X \le m) = 0.5$
   (d) E(X) and Var(X).

8. Given that $f(x) = 0.5$ for $1 \le x \le 2$ and $3 \le x \le b$ is the probability density function of X of a random variable X, find: (a) the value of b
   (b) $P(X \ge 1.5)$
   (c) $P(X \le 3.5 \mid X \ge 1.5)$
   (d) E(X) and Var(X).

9. A continuous random variable has probability density function $f(x) = 0.25$ for $-1 \le x \le 1$ and $3 \le x \le a$. Find:
   (a) the value of a
   (b) $P(X < 3.5)$
   (c) $P(X > 0 \mid X < 3.5)$
   (d) m such that $P(X \ge m) = 0.5$.

10. A continuous random variable X has probability density function $f(x) = ax + b$ for $1 \le x \le 4$ and $P(X \le 2) = 1/9$.
    (a) Find the values of a and b.
    (b) Sketch the probability density function of X
    (c) Find m such that $P(X \ge m) = 0.75$.
    (d) Calculate E(X) and Var(X).

11. The probability density function of a random variable X is given by:

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 \le x \le 2 \end{cases}. \text{ Find:}$$

    (a) $P(X > 1.3)$
    (b) $P(X < 1.5)$
    (c) $P(X > 1.3 \mid X < 1.5)$
    (d) k such that $P(X > k) = 0.25$.

12. Given that the probability density function of a continuous random variable X is given by

$$f(x) = \begin{cases} ax & 0 \le x \le 2 \\ 2a & 2 \le x \le 3 \end{cases}:$$

    (a) Find the value of a.
    (b) Sketch $f(x)$.
    (c) Find $P(X \ge k)$ if $2 \le k \le 3$.
    (d) find m such that $P(X \le m) = 0.5$.

13. The life X (in years) of an electronic component has probability density function modelled by $f(x) = a - \frac{1}{2}(x-1)^2$ for $0 \le x \le 2$.
    (a) Find the value of a.
    (b) Sketch the probability density function of X
    (c) Find the *median* for X, that is find m such that $P(X \ge m) = 0.5$.
    (d) Calculate the mean and standard deviation of X.

14. The length X (in cm) of a biological specimen has a probability density function modelled by $f(x) = \begin{cases} ax & 0 \le x \le 1 \\ \frac{a}{2}(3-x) & 1 \le x \le 3 \end{cases}$.
    (a) Find the value of a.
    (b) Sketch the graph of $f(x)$.
    (c) Find the probability that a randomly chosen specimen will have a length exceeding 1 cm given that its length does not exceed 2 cm.
    (d) Find the probability that in a sample of 20 randomly chosen specimens at least 14 have lengths exceeding 1 cm given that their lengths do not exceed 2 cm.

15. The mass X (in kg) of a small mammal has a probability density function modelled by $f(x) = \begin{cases} 2a(1-x) & 0 \le x \le 0.5 \\ 2ax & 0.5 \le x \le 1 \end{cases}$.
    (a) Find the value of a.
    (b) Sketch the graph of $f(x)$.
    (c) Find the probability that a randomly chosen mammal of this species with a mass less than 0.75 kg will have a weight exceeding 0.5 kg.
    (d) Find the *median* mass and the *mean* mass of the mammal.
    (e) Find the probability that in a sample of 20 randomly chosen specimens at least 6 have mass each less than 0.75 kg given that their mass each exceed 0.5 kg

16. The life X (in years) of a brand of electric globe has a probability density function modelled by $f(x) = \begin{cases} kx & 0 \le x \le 1 \\ \frac{k}{x^5} & x > 1 \end{cases}$.
    (a) Find the value of k.
    (b) Sketch the graph of $f(x)$.
    (c) Find the probability that a randomly selected globe of this brand having a life exceeding 0.5 years will have a life exceeding 1 year.
    (d) Find the median "life".
    (e) Find the probability that in a sample of 12 randomly chosen globes with "lives" exceeding 0.5 years no more than 6 will have "lives" exceeding 1 year.

17. A continuous random variable X has probability density function given by

$$f(x) = \begin{cases} \frac{1}{3} & -2 \le x \le -1 \\ x^2 & 0 \le x \le 1. \\ \frac{1}{12} & 1 \le x \le 5 \end{cases}$$

    (a) Sketch the graph of $f(x)$.
    (b) Find $P(X > 0)$.
    (c) Find $P(X < 4)$.
    (d) Find $P(X < 4 \mid X > 0)$.

15.4 Linear Transformations on random variables

- Consider the random variable X with probability mass function $f(x)$ for $a \le x \le b$.
- Let the random variable T = mX + n where m and n are constants.
  - The expected value, or mean for T is given by:

$$\begin{aligned} E(T) &= E(mX + n) \\ &= E(mX) + E(n) \\ &= mE(X) + n \end{aligned}$$

The proof of this is beyond the scope of this book.

  - The variance for T is given by:

$$\begin{aligned} Var(T) &= Var(mX + n) \\ &= Var(mX) + Var(n) \\ &= m^2 Var(X) + 0. \end{aligned}$$

The proof of this is beyond the scope of this book.

- Hence, if random variables T and X are linearly related in the form T = mX + n,
  - E(T) = mE(X) + n • Var(T) = m^2Var(X)
  - Standard deviation for T = $|m|$ × standard deviation for X.

  These results are true for all random variables, discrete and continuous.

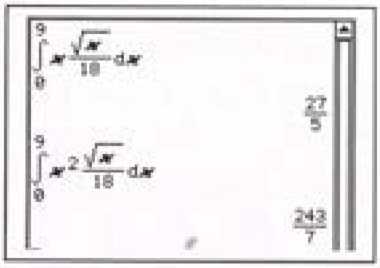
Example 15.6

The random variable X has probability density function $f(x) = \dfrac{\sqrt{x}}{18}$ for $0 \le x \le 9$.

Find the mean μ and standard deviation σ for X and hence find the mean and standard deviation for T if: (a) T = 2X + 5 (b) $T = \dfrac{X-\mu}{\sigma}$.

Solution:

$$\mu = E(X) = \int_0^9 x\frac{\sqrt{x}}{18}\,dx = \frac{27}{5}, \quad E(X^2) = \int_0^9 x^2\frac{\sqrt{x}}{18}\,dx = \frac{243}{7}$$



$$\sigma = \sqrt{E(X^2) - [E(X)]^2} = \sqrt{\frac{243}{7} - \left(\frac{27}{5}\right)^2} = \frac{18\sqrt{21}}{35}$$

(a) $E(T) = 2E(X) + 5 = 2 \times \dfrac{27}{5} + 5 = \dfrac{79}{5}$

Standard deviation for T = $2 \times \dfrac{18\sqrt{21}}{35} = \dfrac{36\sqrt{21}}{35}$

(b) $T = \dfrac{X-\mu}{\sigma} = \dfrac{1}{\sigma}X - \dfrac{\mu}{\sigma}$.

$E(T) = \dfrac{1}{\sigma}E(X) - \dfrac{\mu}{\sigma} = \dfrac{1}{\sigma} \times \mu - \dfrac{\mu}{\sigma} = 0$

Standard deviation for T = $\dfrac{1}{\sigma}$ × standard deviation for X

$= \dfrac{1}{\sigma} \times \sigma = 1.$

If μ and σ are respectively the mean and standard deviation for the random variable X, then the random variable $\frac{X-\mu}{\sigma}$ will have a mean of 0 and a standard deviation of 1.

Exercise 15.3

1. The random variable X has probability density function $f(x) = \dfrac{x}{50}$ for $0 \le x \le 10$.

   Define random variable $T = X + 5$. The probability density function for T is given by $f(t) = \dfrac{t-5}{50}$ for $5 \le t \le 15$.

   (a) On the same set of axes, sketch the probability density functions of X and T.
   (b) Find $P(0 \le X \le 5)$ and $P(5 \le T \le 10)$. (c) Find E(X) and E(T).

2. The random variable X has probability density function $f(x) = 0.1 + 0.04x$ for $0 \le x \le 5$. Define random variable $T = 2(X - 1)$. The probability density function for T is given by $f(t) = 0.07 + 0.01x$ for $-2 \le t \le 8$.

   (a) On the same set of axes, sketch the probability density functions of X and T.
   (b) Find $P(X \le 2 \mid X \ge 1)$ and $P(T \le 2 \mid T \ge 0)$. (c) Find Var(X) and Var(T).

3. The random variable X has probability density function $f(x) = \dfrac{1}{3}(x-1)^2$ for $0 \le x \le 3$.

   Define random variable $T = 4 - 2X$. The probability density function for T is given by $f(t) = \dfrac{1}{6}(1 - 0.5t)^2$ for $-2 \le t \le 4$.

   (a) On the same set of axes, sketch the probability density functions of X and T.
   (b) Find $P(X \ge 1 \mid X \le 2)$ and $P(T \le 2 \mid T \ge 0)$. (c) Find Var(X) and Var(T).

4. The random variable X has probability density function $f(x) = \dfrac{3}{4}(1 - x^2)$ for $-1 \le x \le 1$.

   Define the random variable $T = \dfrac{X - \mu}{\sigma}$ where μ and σ are respectively the mean and standard deviation for the random variable X. The probability mass function for T is given by $f(t) = \dfrac{3\sqrt{5}}{20}(1 - \dfrac{t^2}{5})$ for $-\sqrt{5} \le t \le \sqrt{5}$.

   (a) Find the mean μ and standard deviation σ for X.
   (b) Use integrals to determine the mean and standard deviation for T.
   (c) Hence, verify that $E(T) = \dfrac{1}{\sigma}E(X) = 0$ and $Var(T) = \dfrac{1}{\sigma^2}Var(X) = 1$.

5. The random variable X has probability density function $f(x) = \dfrac{4 - 2x}{9}$ for $-1 \le x \le 2$.

   Define the random variable $T = \dfrac{X - \mu}{\sigma}$ where μ and σ are respectively the mean and standard deviation for the random variable X. The probability mass function for T is given by $f(t) = \dfrac{2\sqrt{2} - t}{9}$ for $-\sqrt{2} \le t \le \sqrt{2}$.

   (a) Find the mean μ and standard deviation σ for X.
   (b) Use integrals to determine the mean and standard deviation for T.
   (c) Hence, verify that $E(T) = \dfrac{1}{\sigma}E(X) = 0$ and $Var(T) = \dfrac{1}{\sigma^2}Var(X) = 1$.

16 The Uniform Distribution

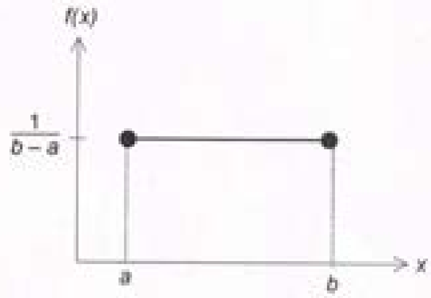
16.1 The Discrete Uniform Distribution

- The discrete uniform distribution was discussed in Chapter 11.

16.2 The Continuous Uniform Distribution

- If the continuous random variable X is uniformly distributed in the interval $a \le x \le b$, then the probability density function of X is given by:

$$f(x) = \begin{cases} \dfrac{1}{(b-a)} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}.$$

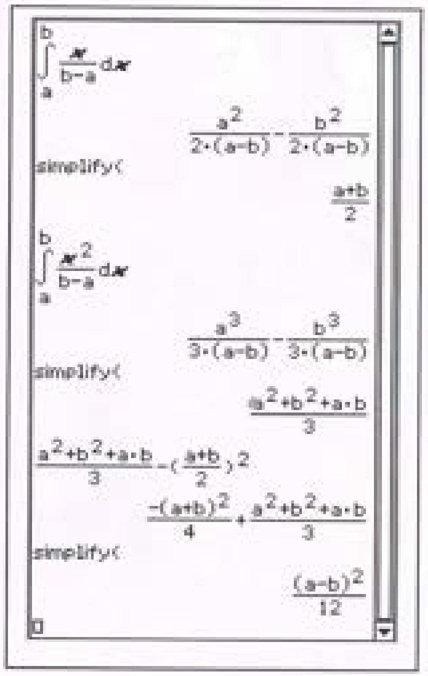


- The graph of $f(x)$ is sketched in the accompanying diagram.
- Note that $f(x)$ is independent of x and is equal to the reciprocal of the width of the interval. The mean value is the midpoint of the interval.
- The mean value or expected value of X,

$$E(X) = \int_a^b x \times \left(\frac{1}{b-a}\right) dx = \frac{a+b}{2}.$$

- The variance of X,

$$\begin{aligned} \text{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= \int_a^b x^2 \times \left(\frac{1}{b-a}\right) dx - \left(\frac{a+b}{2}\right)^2 \\ &= \frac{a^2+b^2+ab}{3} - \left(\frac{a+b}{2}\right)^2 \\ &= \frac{(a-b)^2}{12}. \end{aligned}$$



- Random variables that are equally likely to take any value within a given interval have probability distribution functions that can be modelled by the continuous uniform (rectangular) distribution.
  For example: X: the waiting time for a bus that is scheduled to arrive between 3.30 pm and 3.40 pm .

Example 16.1

The continuous random variable X is uniformly distributed in the interval $3 \le x \le 23$.

(a) Find the probability density function of X.

(b) Sketch the probability density function of X.

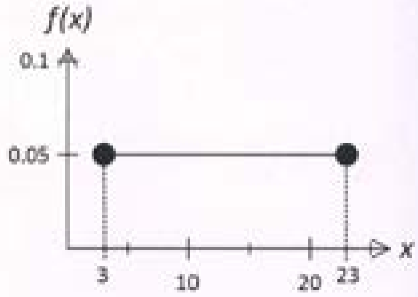
(c) Find P(8 < X < 12). (d) Find P(X < 12 | X > 8)

(e) Find the E(X) and Var(X).

Solution:

(a) Probability density function $f(x) = \begin{cases} \dfrac{1}{20} & 3 \le x \le 23 \\ 0 & \text{otherwise} \end{cases}$.

(b) Sketch of $f(x)$ is shown in the accompanying diagram.



(c) $P((8 < X < 12) = \dfrac{4}{20} = \dfrac{1}{5}$.

(d) $P(X < 12 \mid X > 8) = \dfrac{P(8 < X < 12)}{P(X > 8)}$

$$= \frac{4}{23-8} = \frac{4}{15}.$$

(e) $E(X) = \dfrac{3+23}{2} = 13$. $Var(X) = \dfrac{(23-3)^2}{12} = \dfrac{100}{3}$.

Example 16.2

A continuous random variable X has probability density function $f(x) = 0.2$ for $a \le x \le b$, and $P(X \le 5) = 0.6$. Find:

(a) the values of a and b (b) k such that $P(X \le k) = 0.25$

Solution:

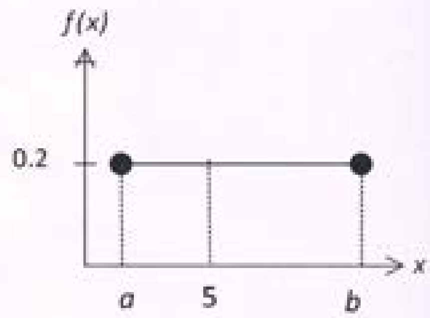
(a) $P(X \le 5) = 0.6 \quad \Rightarrow \quad (5 - a) \times 0.2 = 0.6$

$a = 2$

Since $f(x) = \dfrac{1}{5}$, $b - 2 = 5 \quad \Rightarrow \quad b = 7$

(b) $P(X \le k) = 0.25 \quad \Rightarrow \quad (k - 2) \times 0.2 = 0.25$

$k = 3.25$



Example 16.3

A traffic light has a red cycle that lasts 60 seconds. Jodie arrives at the traffic lights during the red cycle. Define X: waiting time for Jodie at the lights.

(a) Determine the probability density function of X

(b) Sketch the probability density function of X.

(c) Find the probability that Jodie has to wait at least 40 seconds.

(d) Find the probability that Jodie has to wait at least 40 seconds given that she has been waiting at least 30 seconds.

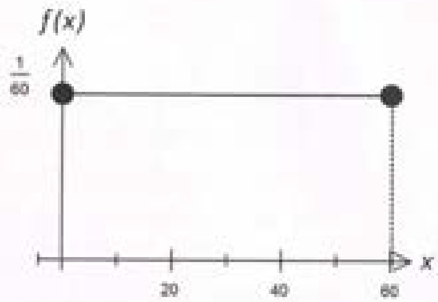
(e) Find Jodie's mean waiting time at the lights and the accompanying standard deviation.

Solution:

(a) Jodie is equally likely to have arrived at the lights during any point of its red cycle. Hence, Jodie is equally likely to have to wait anything between 0 and 60 seconds. Therefore X is uniformly distributed in the interval $0 \le x \le 60$.

The probability density function of X is: $f(x) = \frac{1}{60}$ for $0 \le x \le 60$

(b) The sketch of $f(x)$ is given in the accompanying diagram.



(c) $P(X \ge 40) = \frac{20}{60} = \frac{1}{3}$.

(d) $$P(X \ge 40 \mid X \ge 30) = \frac{P(X \ge 40 \cap X \ge 30)}{P(X \ge 30)}$$
$$= \frac{P(X \ge 40)}{P(X \ge 30)}$$
$$= \frac{(60-40)}{(60-30)} = \frac{2}{3}$$

(e) Mean waiting time = 30 seconds.

Standard deviation = $\sqrt{\frac{(60-0)^2}{12}} = 10\sqrt{3}$ seconds.

Exercise 16.1

1. The continuous random variable X is uniformly distributed in the interval $2 \le x \le 8$.
   (a) Find the probability density function of X. (b) Sketch the pdf of X.
   (c) Find $P(4 < X < 6)$. (d) Find $P(X < 6 \mid X > 4)$.

2. A continuous random variable X is uniformly distributed in the interval $a \le x \le 5a$, $a \ne 0$.
   (a) Find the probability density function of X. (b) Find $P(2a \le X \le 4a)$.
   (c) Find $P(X \ge 2a \mid X \le 4a)$ (d) E(X) and Var(X).

3. A continuous random variable X has probability density function $f(x) = 0.25$ for $1 \le x \le k$. (a) Find the value of k. (b) Find $P(X > 2)$.
   (c) Find $P(X > 2 \cup X < 3)$ (d) Find the mean and variance for X.

4. A continuous random variable X has probability density function $f(x) = 0.1$ for $a \le x \le b$, and $P(X \le 5) = 0.4$. Find:
   (a) the values of a and b (b) k such that $P(X \le k) = 0.25$
   (c) q such that $P(X \le q) = 0.75$ (d) $P(X = 2a)$

5. The pH X of a chemical solution can be assumed to be uniformly distributed in the interval $8 \le x \le 9$.
   (a) Find the probability density function of X.
   (b) Find the probability that a solution of pH greater than 8.2 does not exceed 8.8.
   (c) Find the probability that a solution has a pH less than 8.2 or greater than 8.8.
   (d) Four such solutions were made. Find the probability that:
      (i) none of these solution will have a pH between 8.2 and 8.8.
      (ii) at least one of these solutions will have a pH less than 8.2 or greater than 8.8.

6. The off-cuts of a line of logs processed by a timber yard can be any length between 0 and 2 cm. Define X: length of an off-cut. The probability that an off-cut has a length not exceeding x is kx.
   (a) Find the probability density function of X. (b) Find the value of k.
   (c) Find the probability that an off-cut of at least 0.5 cm is no longer than 1.5cm.
   (d) Ten off-cuts are chosen at random from a processed pile from this yard. Find the probability that exactly five of them have lengths of at least 0.5 cm but not longer than 1.5 cm.

7. A amount of soft drink V dispensed by a dispenser may be any value between 745 mL and 752 mL inclusive.
   (a) Find the probability density function for V.
   State the mean and standard deviation for V.
   (b) Find the probability that a child using this dispenser gets:
      (i) exactly 750 mL of soft drink (ii) less than 750 mL of soft drink.
   (c) Six cups of soft drink were obtained from this dispenser. Find the probability that no more than 5 of these cups had drinks less than 750 mL.

8. The time T to download a movie from an Internet site between 7pm and 10pm daily is any value between 10 minutes and 40 minutes inclusive.
   (a) Find the probability density function of T.
   (b) Find the probability that the time to download the movie is:
   (i) 15 minutes (ii) is more than 20 minutes.
   (c) Find the average time required to download the movie and its associated standard deviation.
   (d) The movie was downloaded on three separate evenings between 7pm and 10pm, find the probability that on at least two of these occasions, it took more than 20 minutes.

9. The red cycle of a traffic light is of duration 120 seconds. Emily arrives at the traffic light when the lights are red. Define T: waiting time for Emily at these lights.
   (a) Find the probability density function for T.
   (b) Find the probability that Emily waits for exactly 1 minute at the lights.
   (c) Find the probability that Emily waits for between 30 and 90 seconds inclusive.
   (d) On eight separate occasions, Emily arrives at this traffic light during its red cycle. Find the probability that on at least 5 occasions she has to wait between 30 and 90 seconds inclusive.

10. A group of year seven students were each asked to draw a circle of radius 10 cm using compasses. Assume that the radius of the circles drawn can have any value between 9.9 cm and 10.1 cm inclusive. Define A: area of circle drawn.
   (a) Find the probability density function for A.
   (b) Find the probability that a randomly chosen circle from this group has an area less than 310 cm^2.
   (c) Find the probability that of three circles randomly chosen from this group at least one will have an area less than 310 cm^2.

11. The continuous random variable X is uniformly distributed in the interval $0 \le x \le 10$.
   (a) Find the mean and standard deviation for the variable $\frac{X-\mu}{\sigma}$ where μ and σ are respectively the mean and standard deviation for X.
   (b) Find $P(\mu - \sigma \le X \le \mu + \sigma)$. (c) Find $(-1 \le \frac{X-\mu}{\sigma} \le 1)$.

12. The continuous random variable X is uniformly distributed in the interval $-6 \le x \le 6$.
   (a) Find the mean and standard deviation for the variable $\frac{X-\mu}{\sigma}$ where μ and σ are respectively the mean and standard deviation for X.
   (b) Find $P(X \le 3\sqrt{3} \mid X \ge 2\sqrt{3})$. (c) Find $(\frac{X-\mu}{\sigma} \le 1.5 \mid \frac{X-\mu}{\sigma} \ge 1)$.

17 The Normal Distribution

17.1 The Probability Density Function of a Normal Distribution

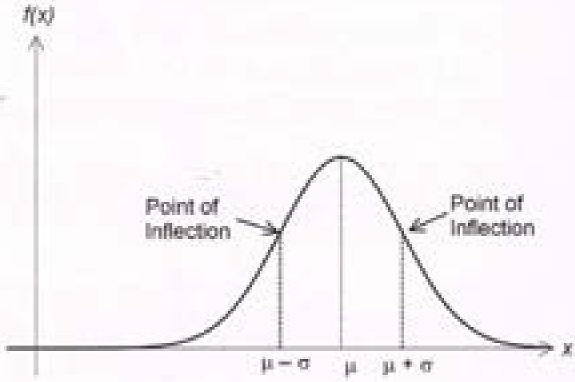
- If X is a continuous random variable with probability density function given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]} \quad \text{for } -\infty < x < \infty,$$

  where μ and σ are positive constants,
  then, X is a *normal* variable with a mean of μ and a standard deviation of σ.
  In symbols, $X \sim N(\mu, \sigma^2)$.

- If $\mu = 0$ and $\sigma = 1$, then, X is a *standard normal* variable, represented by the letter Z. That is, $Z \sim N(0, 1)$

- The graph of the probability density function of a normal variable X with mean μ and standard deviation σ is shown below. The graph of the probability density function is bell-shaped and is symmetrical about the mean μ. The inflection points are located at a distance of σ on either side of the mean. Clearly the larger the value of σ, the "wider" the curve.



17.2 Calculating Normal Probabilities

- To calculate $P(a \le X \le b)$, we need to evaluate $\int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]} dx$.

  The integral however is non-integrable! Hence, to evaluate the value of the integral, numerical methods are used.

- CAS/graphic calculators have built-in routines/programmes that automate the calculation of probabilities associated with the normal distribution.

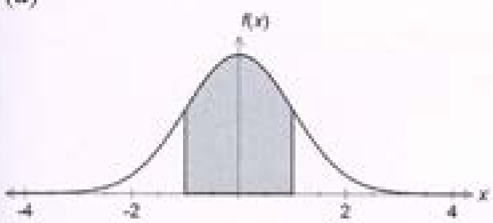
17.2.1 Estimating Probabilities for Normal Variables

- Consider a normal variable X with mean μ and standard deviation σ.
  - Since, the normal curve is symmetrical about the mean μ,
    $P(X \geq \mu) = P(X \leq \mu) = 0.5$.
  - It can be shown that:
    $P(\mu - \sigma \leq X \leq \mu + \sigma) \approx 0.68$
    $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \approx 0.95$
    $P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) \approx 0.997$

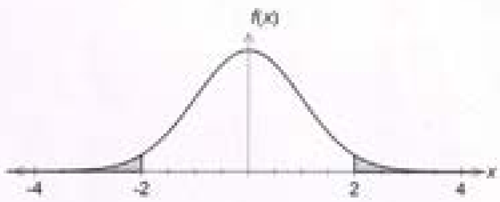
Example 17.1

The diagrams below show the graphs of the probability density functions of the normal variable X with mean 0 and standard deviation 1. Without the use of calculator, estimate the area of the shaded region.

(a)



(b)



Solution:

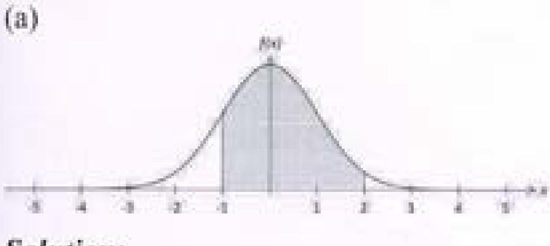
(a) Area of shaded region $= P(-1 \leq X \leq 1)$
≈ 0.68

(b) Area of shaded region
$= 1 - P(-2 \leq X \leq 2)$
$\approx 1 - 0.95 \approx 0.05$

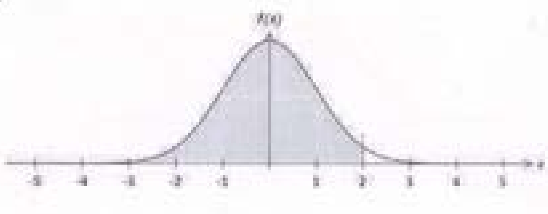
Example 17.2

The diagrams below show the graphs of the probability density functions of the normal variable X with mean 0 and standard deviation 1. Without the use of calculator, estimate the area of the shaded region.

(a)



(b)



Solution:

(a) Area of shaded region
$= P(-1 \leq X \leq 2)$
$\approx \dfrac{0.68}{2} + \dfrac{0.95}{2}$
≈ 0.815

(b) Area of shaded region
$= P(X \leq 2)$
$= 0.5 + \dfrac{0.95}{2}$
≈ 0.975

Example 17.3

Given that X is a normal variable with mean 100 and standard deviation 10, without the use of a calculator, estimate: (a) $P(X \geq 100)$ (b) $P(X \geq 80)$ (c) $P(70 \leq X \leq 120)$.

Solution:

(a) Since, the mean for X is 100, $P(X \geq \mu) = 0.5$

(b) $P(X \geq 80) = P(X \geq \mu - 2\sigma)$

$$\approx \frac{0.95}{2} + 0.5$$
$$\approx 0.975$$

(c) $P(70 \leq X \leq 120) = P(\mu - 3\sigma \leq X \leq \mu + 2\sigma)$

$$\approx \frac{0.997}{2} + \frac{0.95}{2}$$
$$\approx 0.9735$$

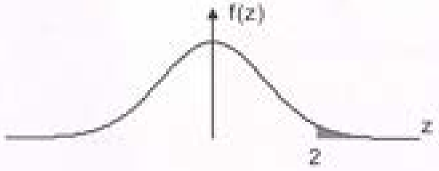
17.2.2 Calculating Probabilities for the Standard Normal Distribution Z ~ N(0, 1)

- In calculating probabilities related to normal variables, we use tables embedded in CAS/Graphic calculators.

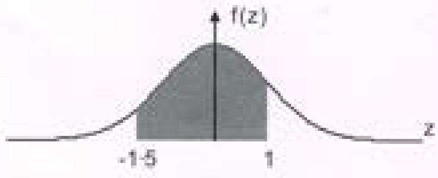
Example 17.4

Find the area of the regions shaded in each of the sketches of the graph of the probability density function of Z ~ N(0, 1):

(a)



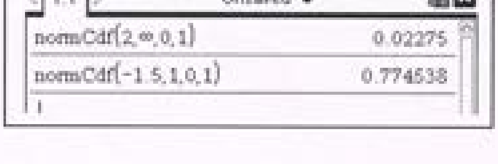
(b)



Solution:

(a) $P(Z \geq 2) = 0.0228$

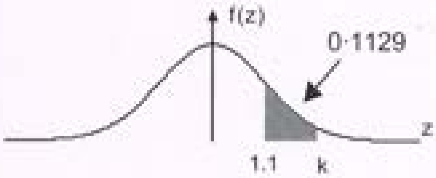
(b) $P(-1.5 \leq Z \leq 1) = 0.7745$



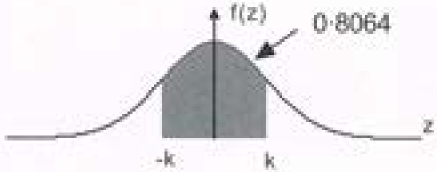
Example 17.5

The diagrams below show the graphs of the probability density functions of the normal variable Z with mean 0 and standard deviation 1. Find the value of k corresponding to the area of each of the shaded regions:

(a)



(b)



Solution:

(a) $P(1.1 < Z < k) = 0.1129$

But $P(1.1 < Z < k) = P(Z < k) - P(Z < 1.1)$.

$\Rightarrow P(Z < k) = 0.1129 + P(Z < 1.1)$

$= 0.1129 + 0.8643 = 0.9772$

Hence, $k = 1.9991 \approx 2.0$

```
normCDf(-∞,1.1,1,0)
                          0.86433
invNormCDf("L",0.9772,1,0)
                          1.99908
```

OR

```
solve(normCDf(1.1,k,1,0)=0.1129,k)
                    {k=1.999705056}
```

$\Rightarrow k = 1.9997 \approx 2.0$

(b) $P(-k \le Z \le k) = 0.8064$

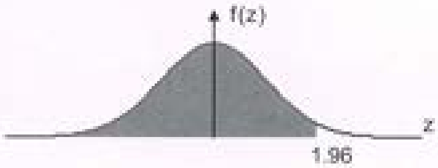
```
solve(normCDf(-k,k,1,0)=0.8064,k)
                    {k=1.300002828}
```

$\Rightarrow k = 1.3$

Exercise 17.1

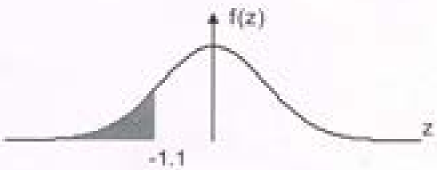
1. Find the area of the regions shaded in each of the sketches of the graph of the probability density functions of $Z \sim N(0, 1)$:

(a)



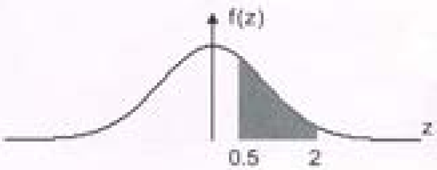
(b)

(c)



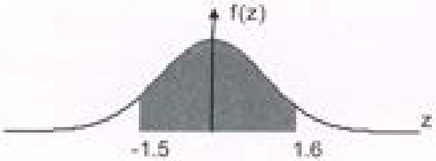
(d)

(e)

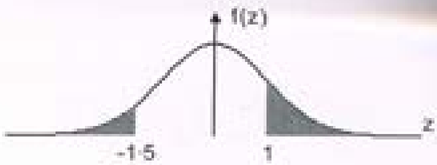


(f)

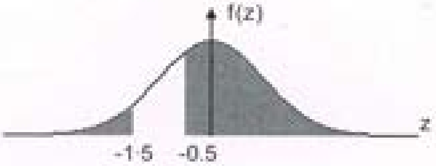
1. (g)



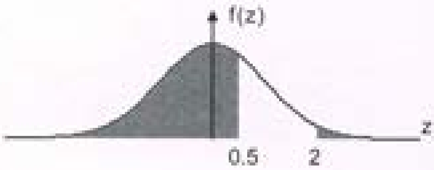
(h)



(i)



(j)



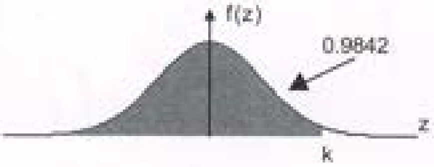
2. Find the value of each of the following, where $Z \sim N(0, 1)$:
   (a) $P(Z < 2.35)$ (b) $P(Z \geq 1.82)$ (c) $P(Z \leq -1.92)$
   (d) $P(Z \geq -1.87)$ (e) $P(1.45 \leq Z \leq 2.89)$ (f) $P(-0.54 \leq Z < 1.76)$

3. Find the value of each of the following, where $Z \sim N(0, 1)$:
   (a) $P(Z \leq 2.02 \cap Z \leq 1.45)$ (b) $P(Z \leq 2.56 \cap Z < -1.25)$
   (c) $P(Z < 0.46 \cup Z > 1.98)$ (d) $P(Z \leq -2.54 \cup Z \geq 2.28)$

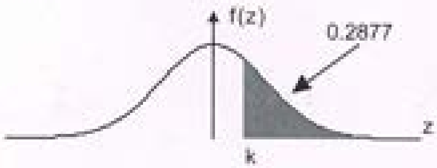
4. Given that $Z \sim N(0, 1)$, find:
   (a) $P(Z \geq 1.45 \mid Z \leq 1.96)$ (b) $P(Z < 0.67 \mid Z \geq -1.56)$
   (c) $P(Z \geq -1.96 \mid Z \leq 1.96)$ (d) $P[(Z > 0.54 \cup Z < -0.54) \mid Z \leq 0]$

5. Given that $Z \sim N(0, 1)$, find the value of *k* corresponding to the area of each of the shaded regions:

(a)

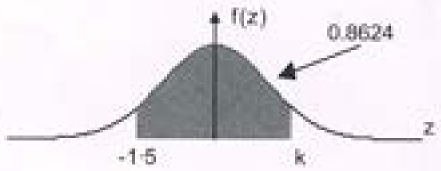


(b)

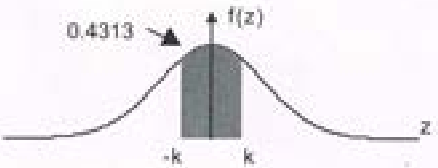


(c)

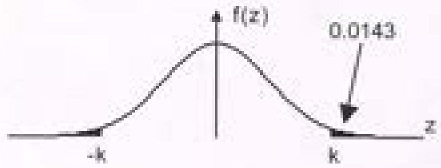
(d)



(e)



(f)



17.2.3 Standardisation of non-standard normal variables

- If X is a normal variable with mean μ and standard deviation σ, then as shown in Section 13.4 of this book, the variable $\frac{X-\mu}{\sigma}$ is a normal variable with a mean of 0 and a standard deviation of 1.
- That is, if $X \sim N(\mu, \sigma^2)$ then $\frac{X-\mu}{\sigma} \sim Z(0, 1)$.

  X which is a normal variable with mean μ and standard deviation σ, has been transformed into a standard normal variable by applying the transform $\frac{X-\mu}{\sigma}$.

  - Hence, $P(a \le X \le b) = P(\frac{a-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma})$

    $= P(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma})$.
  - $\frac{a-\mu}{\sigma}$ and $\frac{b-\mu}{\sigma}$ are respectively the *standard scores* associated with a and b.
  - Standard scores measure the "distance" of a variable from its mean in terms of its standard deviation.

Example 17.6

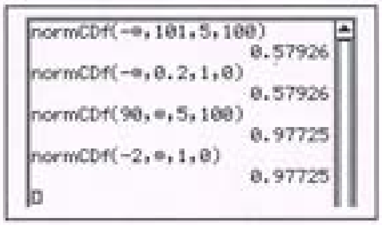
Given that X is a normal variable with a mean of 100 and a standard deviation of 5.
(a) Find k where $P(X \le 101) = P(Z \le k)$. Hence, find $P(X \le 101)$.
(b) Find k where $P(X \ge 90) = P(Z \ge k)$. Hence, find $P(X \ge 90)$.

Solution:

(a) In the expression $P(X \le 101) = P(Z \le k)$,
k is the standard score for 101.

That is, $k = \frac{101-100}{5} = 0.2$.

Hence, $P(X \le 101) = P(Z \le 0.2) = 0.57926$



(b) In the expression $P(X \ge 90) = P(Z \ge k)$,
k is the standard score for 90.

That is, $k = \frac{90-100}{5} = -2$.

Hence, $P(X \ge 90) = P(Z \ge -2) = 0.97725$

Example 17.7

X is a normal variable with a mean of 50 and standard deviation of 7. Find:

(a) $P(X > 55)$ (b) $P(X > 55 \mid X > 45)$.

Solution:

(a) $P(X \geq 55) = 0.23753$.

(b)
$$P(X > 55 \mid X > 45) = \frac{P(X > 55 \cap X > 45)}{P(X > 45)}$$
$$= \frac{P(X > 55)}{P(X > 45)} = \frac{0.23753}{0.76247} = 0.3115.$$

Example 17.8

X is a normal variable with a mean of 120 and a standard deviation of 20. Find k such that:

(a) $P(X \leq k) = 0.75$ (b) $P(120 - k < X < 120 + k) = 0.9$ (c) $P(115 \leq X \leq k) = 0.5$.

Solution:

(a) $P(X \leq k) = 0.75 \quad \Rightarrow \quad k = 133.4898 = 133.5$.

(b) $P(120 - k < X < 120 + k) = 0.9 \quad \Rightarrow \quad 120 - k = 87.10293 \quad \Rightarrow \quad k = 32.9$.

(c) $P(115 \leq X \leq k) = P(X \leq k) - P(X \leq 115)$

Hence, $P(X \leq k) = 0.5 + P(X \leq 115)$

Therefore $P(X \leq k) = 0.5 + 0.40129$

$= 0.90129$

$\Rightarrow k = 145.77874 = 145.8$

```
invNormCDf("L",0.75,20,120)
                     133.48980
invNormCDf("C",0.9,20,120)
                      87.10293
invNormCDf("L",0.90129,20,120▸
                     145.77874
```

OR

```
solve(normCDf(115,k,20,120)=0.5,k)
                 {k=145.7791621}
```

$\Rightarrow k = 145.8$

Example 17.9

Given that $X \sim N(\mu, 16)$, find μ if $P(X \geq 20) = 0.01$.

Solution:

$P(X \geq 20) = 0.01$

Standardise X, $P(Z \geq \frac{20 - \mu}{4}) = 0.01$.

Hence $\frac{20 - \mu}{4} = 2.32635$

Therefore $\mu = 10.69$

```
invNormCDf("R",0.01,1,0)
                     2.32635
solve((20-x)/4=2.32635,x)
                {x=10.69460}
```

OR

```
solve(normCDf(20,∞,4,μ)=0.01,μ)
                {μ=10.6946085}
```

$\Rightarrow \mu = 10.69$

Example 17.10

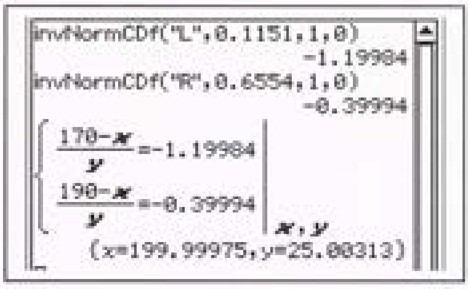
Given that $X \sim N(\mu, \sigma)$, find μ and σ if $P(X \le 170) = 0.1151$ and $P(X \ge 190) = 0.6554$.

Solution:

$$P(X \le 170) = 0.1151$$

Standardise X, $P(Z \le \frac{170-\mu}{\sigma}) = 0.1151$.

Hence $\frac{170-\mu}{\sigma} = -1.19984$ I



$$P(X \ge 190) = 0.6554$$

Standardise X, $P(Z \ge \frac{190-\mu}{\sigma}) = 0.6554$.

Hence $\frac{190-\mu}{\sigma} = -0.39994$ II

Solve I & II simultaneously:

$$\mu = 200 \text{ and } \sigma = 25.$$

Exercise 17.2

1. X is a normal variable with mean 10 and standard deviation 2.
   (a) Find k such that $P(X \ge 12) = P(Z \ge k)$. Hence, find $P(X \ge 12)$.
   (b) Find k such that $P(X \ge 16) = P(Z \ge k)$. Hence, find $P(X \ge 16)$.

2. X is a normal variable with mean 50 and variance 25.
   (a) Find k such that $P(X \le 55) = P(Z \le k)$. Hence, find $P(X \le 55)$.
   (b) Find k such that $P(X \ge 45) = P(Z \ge k)$. Hence, find $P(X \ge 45)$.

3. Given that X is a normal variable with mean 185 and variance 169, find:
   (a) $P(X \le 180)$ (b) $P(X \ge 170)$ (c) $P(X \le 180 \mid X \ge 170)$

4. Given that X is a normal variable with mean 16 and standard deviation 3, find:
   (a) $P(X \ge 20 \cup X \le 14)$ (b) $P(X \le 10 \cup X \ge 22)$ (c) $P(|X - 16| \le 6)$

5. For $Z \sim N(0, 1)$, find the value of k in each of the following:
   (a) $P(Z \le k) = 0.1234$ (b) $P(Z \ge k) = 0.5672$
   (c) $P(0.56 \le Z \le k) = 0.1589$ (d) $P(k \le Z \le 1.98) = 0.7851$
   (e) $P(-k \le Z \le k) = 0.8132$ (f) $P(-k \le Z \le k) = 0.4524$

6. Given $X \sim N(80, 100)$, find k such that:
   (a) $P(X \le k) = 0.95$ (b) $P(X \le k) = 0.05$
   (c) $P(80 \le X \le k) = 0.25$ (d) $P(k \le X < 80) = 0.40$
   (e) $P(80 - k \le X \le 80 + k) = 0.5$ (f) $P(80 - k < X < 80 + k) = 0.995$

7. Find μ if: (a) $P(X \le 20) = 0.1587$ where $X \sim N(\mu, 25)$
   (b) $P(\mu \le X \le 20) = 0.3413$ where $X \sim N(\mu, 100)$.

8. Find σ if: (a) $P(X \ge 75) = 0.2023$ where $X \sim N(65, \sigma^2)$
   (b) $P(50 \le X \le 60) = 0.2995$ where $X \sim N(55, \sigma^2)$.

9. Given that $X \sim N(\mu, \sigma^2)$, find μ and σ if:
   (a) $P(X \ge 70) = 0.1817$ and $P(X < 80) = 0.9655$
   (b) $P(X \ge 167) = 0.3085$ and $P(|X - \mu| \le 10) = 0.2313$.

17.3 Applications using the Normal Distribution

Example 17.11

The marks (out of 100) for an Engineering Examination taken by 200 students may be assumed to be normally distributed with mean 58 and standard deviation 14.2. Students with marks two standard deviations above the mean are awarded "distinctions" while students with marks two standard deviations below the mean are awarded "fails".

(a) Find the number of "distinctions" and "fails" awarded.
(b) Find the probability that a student randomly chosen from this group is awarded a distinction given that the student passed the examination.
(c) The top 0.5% of students are awarded "high distinctions". Find the cut-off mark for the award of a "high distinction".

Solution:

Let X: Marks for the Engineering Examination. $\Rightarrow$ $X \sim N(58, 14.2^2)$.

(a) The proportion of students awarded distinctions $= P[X > 58 + 2(14.2)]$
$= 0.02275$
Hence, the number of distinctions awarded $= 200 \times 0.02275 \approx 4$.

(b)
$$P(\text{student gets a distinction} \mid \text{student passed}) = P(Z > 2 \mid Z \ge -2) = \frac{P(Z > 2)}{P(Z \ge -2)} = \frac{0.02275}{0.97725} = 0.0233$$

(c) Let the cut-off mark for a high distinction be k.
Then
$$P(X \ge k) = 0.005$$
$$P(Z \ge \frac{k-58}{14.2}) = 0.005$$
$$\frac{k-58}{14.2} = 2.57583$$
$$k = 94.6$$

Exercise 17.3

1. The length of screws in a given batch is normally distributed with a mean of 10 mm and a standard deviation of 1.1 mm. A screw is randomly selected from this batch. Find the probability that the length of the screw:
   (a) exceeds 9.9 mm
   (b) does not exceed 10.2 mm
   (c) does not exceed 10.2 mm given that it exceeds 9.9 mm.

2. The diameter of cylindrical rods in a given batch is normally distributed with a mean of 20 mm with a standard deviation of 1.5 mm. Find the probability that a diameter of a randomly chosen rod from the batch:
   (a) does not exceed 21 mm
   (b) is between 18 mm and 21 mm
   (c) exceeds 18 mm given that it does not exceed 21 mm.

3. The time to complete a 50 metre sprint for children of a certain age group is normally distributed with a mean of 12 seconds and a standard deviation of 1.9 seconds. Children with completion times exceeding the mean by more than three standard deviations are referred to the school nurse. Find the probability that a randomly chosen child from this group:
   (a) will have a completion time that exceeds the mean by more than three standard deviations.
   (b) is a candidate for referral to the school nurse
   (c) is a candidate for referral to the school nurse given that the completion time is more than two standard deviations from the mean.

4. The marks (out of 100) of a Mathematics Competition taken by 10 000 students is normally distributed with a mean of 62 with a standard deviation of 13.6.
   (a) The top 0.5% of students are awarded Certificates of High Distinction. Find the minimum mark (to the nearest mark) required to receive a Certificate of High Distinction.
   (b) Certificates of Distinction are to be awarded to the next 10% of students. Find the interval of marks within which a student is required to achieve to receive a Certificate of Distinction.
   (c) Students with marks less than 40 are awarded Certificates of Participation. How many students will be awarded Certificates of Participation.

5. The life of a brand of electric light globes is normally distributed with a mean of 1000 hours and a standard deviation of 120 hours. Globes with lives of less than 700 hours are rejected and recycled.
   (a) Find the proportion of globes that are rejected and recycled.
   (b) In an effort to reduce the proportion of globes that are rejected, the quality control processes are tightened with the aim of altering the standard deviation of the life of the globes (mean unchanged) so that the proportion of globes that are rejected is 0.1%. Find the standard deviation required to achieve this.

6. The weight of an adult rodent is normally distributed with a mean of 50g and a standard deviation of 3.6g. In a field study, an adult rodent was trapped and weighed.
   (a) Find the probability that the weight of this adult rodent is less than 55g.
   (b) Find the probability that the weight of this adult rodent differs from the mean by no more than 5g.
   (c) Find the probability that the weight of this adult rodent differs from the mean by no more than 5g given that its weight is less than 55g.
   (d) Three other adult rodents were trapped and weighed. Find the probability that at least one of the three rodents weighs less than 50g.

7. The speed of vehicles on a freeway is a normal variable with a mean 90 kmh^{-1} and standard deviation 15 kmh^{-1}. On the freeway, vehicles must have speeds of at least 80 kmh^{-1} but not more than 100 kmh^{-1}.
   (a) Find the proportion of vehicles that have speeds below the permissible range.
   (b) Find the proportion of vehicles that have speeds outside the permissible range.
   (c) Five cars travelling on the freeway were radar-gunned to determine their speeds. Find the probability that all 5 cars had speeds within the permissible range.
   (d) Cars with speeds 10% above the maximum speed limit are picked up for "speeding". Find the proportion of cars that will be liable for speeding tickets.
   (e) Cars with speeds less than 90% of the minimum speed are picked up for "obstructing the free flow of traffic". Find the proportion of cars that are in this category.

8. The lengths of a supply of metal stakes are normally distributed with mean 1 metre and standard deviation 3 cm. A stake with length less than 97 cm is rejected and recycled. A stake of length between 97 cm and $100 + k$ cm, inclusive, is filed down to a length of 97 cm. A stake with length exceeding $100 + k$ cm has a piece of length $2 + k$ cm cut from it and it is then filed down to a length of 97 cm. 500 stakes were produced.
   (a) Find the number of stakes that will be rejected.
   (b) Find the number of non-rejected stakes that have to be cut if $k = 4$.
   (c) Find the value of k so that there is only 80% of non-rejected stakes that have to be filed.
   (d) Find the probability that in a sample of 5 stakes, exactly two is rejected.

9. The length of time Jane takes to complete her homework (excluding in between rest breaks) at the end of each school day is normally distributed with mean 2 hours and standard deviation 10 minutes. Each day after school she starts her homework session at 7.30 pm and takes a 10 minute rest break at the end of each hour. She takes no other breaks.
   (a) Find the probability that the length of time Jane takes to complete her homework (excluding rest breaks) is not more than 1 hour and 50 minutes.
   (b) Find the probability that Jane completes her homework by 9.30 pm.
   (c) Find the probability that Jane competes her homework by 10.00 pm.
   (d) Find the probability that Jane completes her homework after 9.30 pm given that she finishes her homework by 10.00 pm.
   (e) Find the probability that on five weekdays, Jane completes her homework by 9.30 pm on exactly 2 days.

10. The diameter of a brand of reticulation hose (sold in lengths of 20m) is normally distributed with mean 10 mm. Find the standard deviation of the diameter of the hose if:
    (a) 10% of these hoses have diameters exceeding 10.5 mm
    (b) 10% of these hoses have diameters less than 9.8 mm
    (c) 95% of these hoses have diameters that differ from the mean by less than 0.2 mm.

11. The marks (out of 180) for a Mathematics Examination taken by 5 000 students is normally distributed with a mean of 102. Find the standard deviation of the marks if:
    (a) 10 students scored above 178 marks (b) 30 students scored below 10 marks
    (c) 40% of the students had marks within 20 marks from the mean.

12. The weight of a batch of organically grown tomatoes is normally distributed. Find the mean and standard deviation for the weight if:
    (a) 15.87% of the tomatoes have weights that exceed the mean by 8g and 73.40% of tomatoes have weights less than 100g
    (b) 68.27% of tomatoes have weights within 6g of the mean and 4.78% of tomatoes have weights less than 140g.

13. The marks (out of 100) for an English Examination are assumed to be normally distributed. Find the mean and standard deviation for the marks if:
    (a) 17.24% of students had scores above 70 and 0.25% of students had scores above 95
    (b) 2.41% of students had scores above 80 and 31.07% of students scored less than 40

14. The power consumption of a certain brand of light bulb is rated at 100 Watts and may be assumed to follow a normal distribution with mean 100 W and standard deviation 2 W.
    (a) Calculate the probability that a randomly chosen bulb will have a power consumption
        (i) exceeding 101 Watts (ii) between 98 and 101 Watts
        (iii) less than 105 Watts given that its power consumption exceeds 101 Watts
    (b) Find k if the probability of a bulb with a consumption exceeding k Watts is 10%.
    (c) Find the probability that in a sample of 3 such bulbs, exactly 1 bulb will have a power consumption of between 98 and 101 Watts.

15. The daily consumption of electricity per house in St. Ian's Mews with 50 similar houses has a normal distribution with mean of 8 units and standard deviation of 2 units.
    (a) Find the probability that a house selected at random uses not less than 4 units and not more than 11 units in one day.
    (b) Find the probability that a house selected at random from those houses with consumption between 4 and 11 units inclusive, has a consumption of less than 10 units.
    (c) Find the daily consumption which is exceeded by 5% of the houses.
    (d) Find the probability that on a given day exactly 40 houses have a consumption of between 4 and 11 units.

18 Sampling

18.1 Samples

- Consider the scenario where the Health Department of a state wishes to know the proportion π of the 100 000 high school students in the state who are short-sighted.
- To determine the true value for π, each and every student in the state would have to be tracked down. The department could then:
  - require each student to complete a questionnaire regarding their eyesight
  - and/or require each student to have their eyes tested.
- This procedure is called a census. As can be imagined, this project would have been very labour and time intensive and costly.
- A far more efficient approach would be to test a sample of students from across the state and from this sample, form a reliable estimate for π.
- For the estimate to be reliable, the members of this sample must be carefully chosen and must reflect the different "types" of students in the state.
- In statistical terms, the samples must be random and unbiased.
- A sample that consists predominantly of students from one ethnic group or socio-economic group or from one geographical locality would be unrepresentative of students across the state. Such a sample may not provide a reliable estimate for π.

18.1.1 Simple random samples

- Consider again the group of 100 000 high school students. A sample of 200 students is to be selected from this group. Each of the 100 000 students is given a unique number from 000 001 to 100 000.
- Two hundred numbers from 000 001 to 100 000 are then drawn either from a "barrel" or using an electronic random number generator. In this instance, each of the 100 000 students has an equal chance of being selected.
- Such a sample is called a ***simple random sample***. That is, in a simple random sample, at the *start* of the process, every member from the population (base set) has an equal chance of being chosen.
- The selection may be done with replacement or without replacement. If the selection is done with replacement, then, if the same number is selected more than once, it is just ignored. Note that if the selection is done without replacement, then the selection process is not independent.
- However, because of random effects, we could end up with a sample of 200 that consists of students from one predominant sex or age group. In which case, the sample would no longer be representative of the population.

18.1.2 Systematic or interval samples

- Consider again the group of 100 000 high school students. A sample of 200 students is to be selected from this group. Each of the 100 000 students is given a unique number from 000 001 to 100 000.
- Instead of choosing two hundred numbers from 000 001 to 100 000 as in the case of a simple random sample, one number is chosen randomly from 000 001 to 000 500.
- If for example, the random number 314 is chosen. Then, 314 and every 500th number following 314 is chosen to form the sample of 200 numbers.
- As in the case of a simple random sample, at the *start* of the sampling process, each student has the same chance of being chosen. Note again that the selection process is not independent. Once, the first number is chosen, the other numbers forming the sample are automatically selected and those outside the interval have no longer any chance of being selected.

18.1.3 Stratified samples

- In strata sampling, the population is divided into several strata (layers). The choice of strata is determined from prior knowledge of the composition of the population.
- In the case of the 100 000 high school students, if the Health Department is aware that short sightedness is correlated to ethnic groups, then one of the strata or sub-strata would be the ethnicity of these students.
- As another example, if year level is a suspected variable in the proportion of short-sighted students, then the entire population would be divided into year levels. Further, if the ratio of the year 7: year 8: year 9: year 10: year 11: year 12 students is determined to be 3: 3: 3: 2: 2, then the composition of members in the sample of 200 should also reflect this ratio. There should then be approximately 46 year 7 students in the sample. These 47 students may be selected using simple random sampling or systematic random sampling.
- In stratified samples, not all members of the population have an equal chance of being selected. But members within the same stratum should have the same chance of being chosen.
- Clearly, constructing stratified samples can be both time consuming, labour intensive and expensive.

18.1.4 Cluster samples

- In stratified samples, it is possible that members of a particular stratum may reside in remote and not easily accessible locations. To avoid this complication, the year groups could be further divided into several clusters representing students from major towns. Then several of these clusters are selected to make up the representatives of the year level. All members of the selected cluster are selected and no students outside these selected clusters are selected.

18.1.5 Convenience samples

- In convenience sampling no regard is paid to the need for the sample to be representative of the population. If a sample of 200 students is required, then a possible convenience sample would the first 200 students in a high school close to where the researcher lives.

18.1.6 Quota samples

- In quota sampling, the population is divided into various strata as in the case of stratified randomly sampling. However, in place of a simple random sampling method of selecting the students in a given stratum, a method of convenience sampling is employed.

18.1.7 Self-selection samples

- In self-selection or volunteer samples, the members of the population volunteer themselves rather than being selected. Hence, if 200 students are required, the first two hundred students to respond positively to a mass email send-out by the Health Department would constitute the sample.

18.2 Random and non-random samples

- A sample is termed *random* if every member of the population from which the sample is derived has a known non-zero probability of being selected. These probabilities need not be the same for each member of the population.
- Hence, simple random samples, systematic samples, stratified samples, cluster samples are random samples.
- However, convenience samples, quota samples and self-selection samples are non-random samples as some members of the population have a zero chance of being selected.

18.3 Bias

- Bias is a statistical concept and is not a reflection of the statisticians' intent to mislead/deceive.
- A sample is termed *biased*:
  - if some of the members of the population from which the sample is derived: has either a zero chance of being selected
  - or the probability of these members being selected cannot be determined before-hand.
- Hence, non-random samples are by definition most likely to be biased. Most samples are biased to some degree as it is almost impossible to design and conduct a bias-free sample.

18.3.1 Sources of Bias and Reducing Bias

- In designing a sampling method, costs, efficiency and the reduction of bias are key factors. The quickest and cheapest sampling method may be more prone to bias while a sample less prone to bias could be prohibitively expensive and time consuming to construct.
- If estimates of a population parameter are required in a hurry, then the bias factor while still important would have to recede into the background.
- Stratified random sampling would be the best sampling method in terms of bias reduction. However, the various strata need to be clearly and correctly identified. The use of wrong variables to describe the strata may lead to unreliable results or unnecessary waste of finances.

Example 18.1

In a survey conducted by a Primary School, 240, 90, 170 students respectively indicated that they walked/rode their bicycles, took a train/bus or were driven to school. 50 students are to be selected to attend a road-safety competition.
(a) How many students from those who were driven to school ought to be selected?
(b) What would be a fair way of selecting the students in (a)?

Solution:

(a) Proportion of students driven to school $= \dfrac{170}{240+90+170} = 34\%$

Hence, required number of students in sample $= 50 \times 34\% = 17$.

(b) The use of a simple random sampling method.
Each of the 170 students driven to school would have their names written on a piece of paper and placed in a barrel. A student or staff-member could then be asked to randomly draw 17 names from the barrel.

Example 18.2

Consider the scenario where the Health Department of a state wishes to know the proportion π of the 100 000 high school students in state who are short-sighted. In each of the following cases, comment on the sampling method used and discuss possible sources of bias.

(a) A health department official visits the high school located 200 m from where he lives and waits at the bus stop just outside the school half an hour before the start of school. He observes the first 100 students that alight from the school buses and notes that 15% of these students wear glasses. From this he estimates that 15% of high school students in the state are short-sighted. Comment on the sampling method used.

(b) Another health department official visits the high school located 2 km from where he works and waits at the drop off point at the school entrance half an hour before the start of school. He observes the first 100 students that are dropped off by cars and notes that 65% of these students wear glasses. From this he estimates that 65% of high school students in the state are short-sighted.

(c) Another health department official emails all high school students enrolled in state schools. The emails requests 500 students to volunteer (by return email) for free eye-tests and offers free glasses for those that found to require them. It was found that 73% of the respondents were short-sighted.

Solution:

(a) This an example of convenience sampling. It is quick, efficient and cheap but non-random and hence is prone to be heavily biased.
The official assumed that the wearing of glasses is evidence of short-sightedness. There are other vision defects that require the wearing of glasses.
It is also possible that students who take buses to school are those that are not short sighted.

(b) This an example of convenience sampling. It is quick, efficient and cheap but non-random and hence is prone to be heavily biased.
The official assumed that the wearing of glasses is evidence of short-sightedness. There are other vision defects that require the wearing of glasses.
It is also possible that students who are dropped off to school are short-sighted as parents are more concerned with their safety and well-being.

(c) This an example of self-selection sampling. It is quick, efficient and cheap but non-random and hence is prone to be heavily biased.
The emails were sent only to students of state schools. Students attending private schools were automatically ruled out.
The high proportion of students who responded to the survey could be due to the offer of free-glasses and hence only students who are already short-sighted have responded.

Exercise 18.1

1. On a particular media website, readers were invited to respond to the following question: Do you agree that asylum seeker boats should be forcefully turned away from Australian territorial waters? Comment on this method of sampling public opinion on the matter and discuss possible sources of bias.

2. Following the release of the latest federal budget, the Australian public was invited to "twitter" their favourable or non-favourable responses. Comment on this method of sampling for gauging the proportion of the Australian public that found the budget favourable and discuss possible sources of bias.

3. On a popular celebrity cooking show, viewers were invited to *sms* their choice of one of two celebrities for the title of "Australia's favourite celebrity chef". Comment on this method of polling for the "title" and discuss possible sources of bias.

4. A fast food chain recently released a new low-calorie budget menu. The Australian public were invited to register their "likes" or "dislikes" on the company's Facebook page. Comment on this method of sampling the public's response to the new menu and discuss possible sources of bias.

5. A city council's cost cutting measures include the switching off of street lamps between the hours of 2 am and 6 am each night. A street with 300 dwellings fronting the street was randomly selected and a method of systematic sampling was used to select 30 homes from this street. Comment on the choice of sampling method and discuss possible sources of bias and ways to reduce them.

6. A radio station conducted on phone poll of 500 homes in a large city to determine the level of support to the issue of daylight saving. Comment on the choice of sampling method and discuss possible sources of bias.

7. An expensive private school (single sex) conducted a phone poll on the families of 200 of its students to gauge the proportion of families supporting the extending of school hours for the students. Comment on the choice of sampling method and discuss possible sources of bias and ways to reduce them.

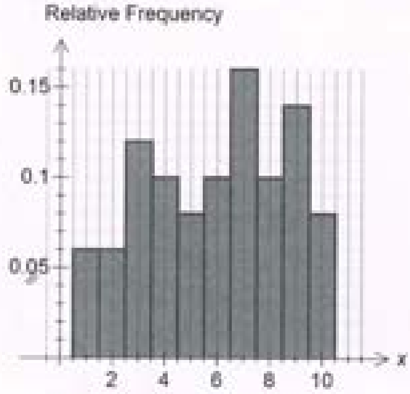
8. A co-educational senior college has 380 year 10, 320 year 11 students and 360 year 12 students. The staff association of the college recommends that recess should be shortened by 5minutes and lunch by 10 minutes. Construct a sample of 100 students to collect information on the students' response to this recommendation addressing possible sources of bias and procedures of reducing these biases.

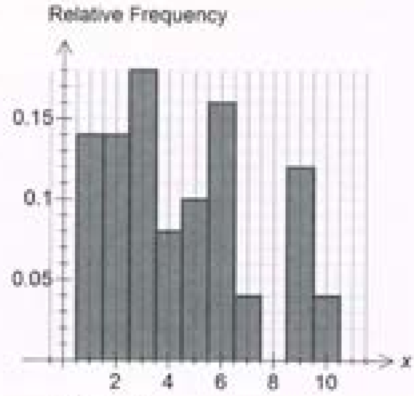
9. A sample of 1000 year 8 students from a city is to be selected to participate in an international mathematics test. Suggest a method to select this sample addressing possible sources of bias and procedures of reducing these biases.

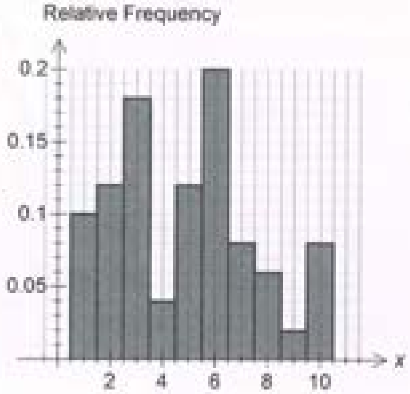
10. Suggest a method to select 3000 Australian citizens to gauge their opinion as to whether Australia needs a new national flag. Address possible sources of bias and your efforts to reduce them.

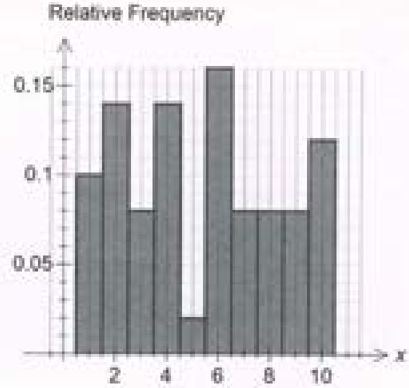
18.4 Variability of random samples.

- The accompanying diagrams show four histograms representing four different samples of fifty observations each from a discrete uniform variable X with probability mass function $f(x) = \dfrac{1}{10}$ for $x = 1, 2, 3, \ldots, 10$.

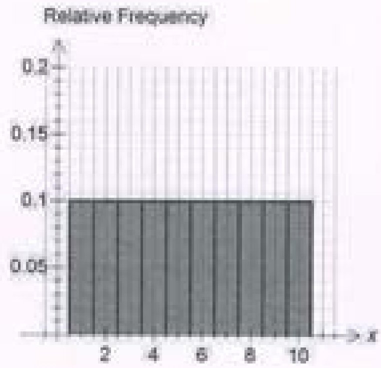






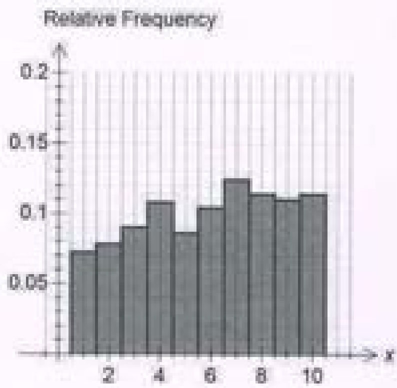


- Theoretically, the histogram for a sample of 50 observations of X should look like the histogram on the right.

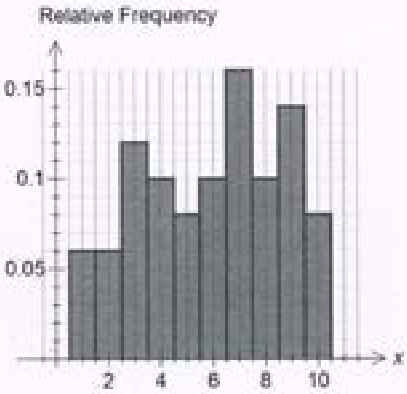


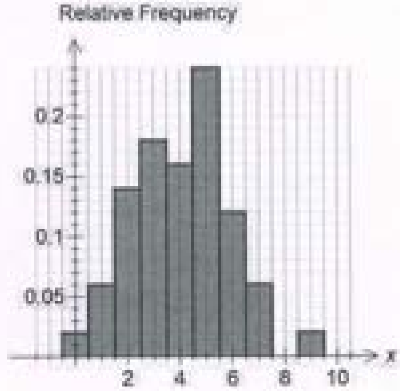
- However, due to the effects of randomness, the histograms are as shown above. This is not in any way evidence of bias in the sampling process.

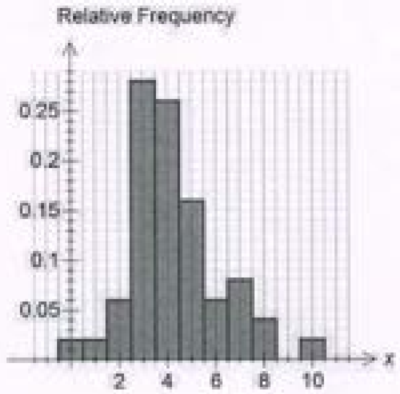
- However, as the sample size increases, the histograms should more resemble the theoretical histogram as seen in the accompanying diagram; on the right; sample size = 500.

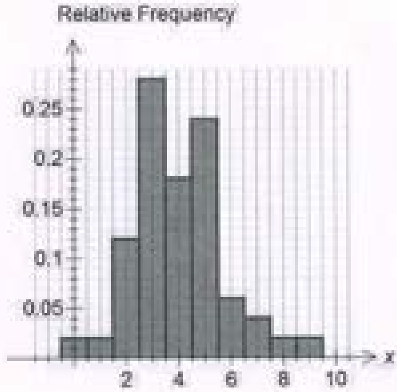


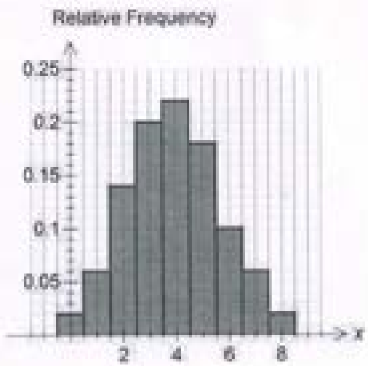
- The accompanying diagrams show four histograms representing four different samples of fifty observations each from a Binomial variable X with parameters $n = 20$ and $p = 0.2$.



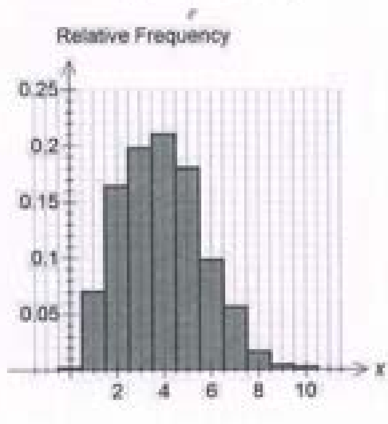






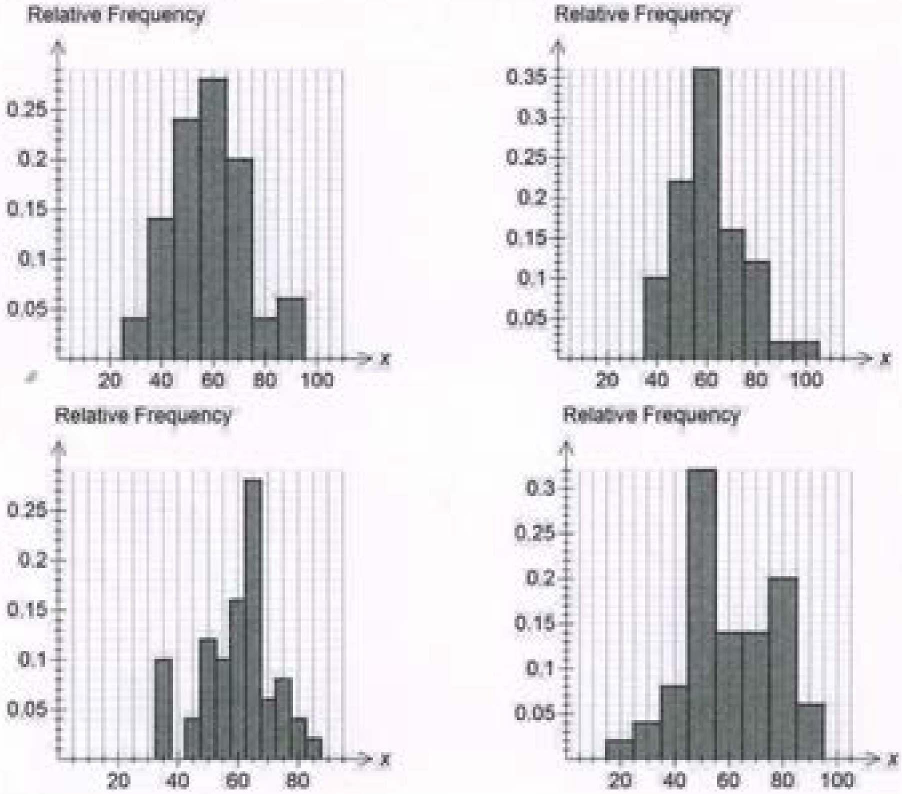


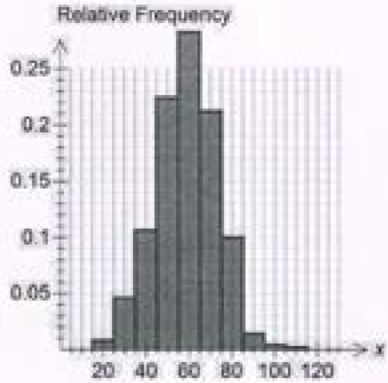
- Theoretically, the histogram for a sample of 50 observations of X should look like the histogram in the accompanying diagram on the right.
- However, due to the effects of randomness, the histograms are as shown above. This is not in any way evidence of bias in the sampling process.



- However, as the sample size increases, the histograms should more resemble the theoretical histogram as seen in the accompanying diagram on the right; sample size = 500.

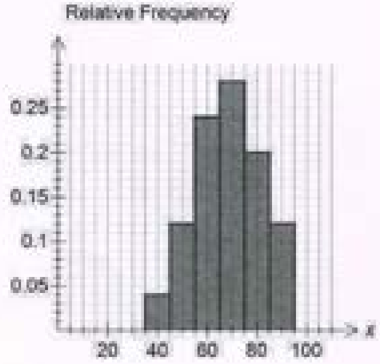
- The accompanying diagrams show four histograms representing four different samples of fifty observations each from a Normal variable X with parameters $\mu = 58$ and $\sigma = 14$.





- Theoretically, the histogram for a sample of 50 observations of X should look like the histogram in the accompanying diagram on the right.

- However, due to the effects of randomness, the histograms are as shown above. This is not in any way evidence of bias in the sampling process.

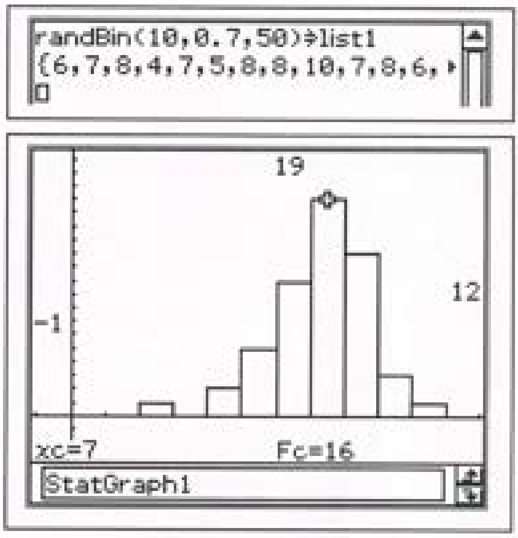


- However, as the sample size increases, the histograms should more resemble the theoretical histogram as seen in the accompanying diagram on the right; sample size = 500.

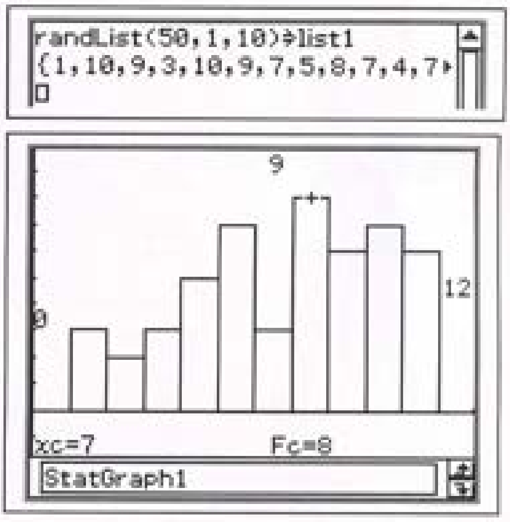
Hands On Task 18.1

In this task, we will make use of a CAS calculator to generate random samples consisting of observations from several random variables.

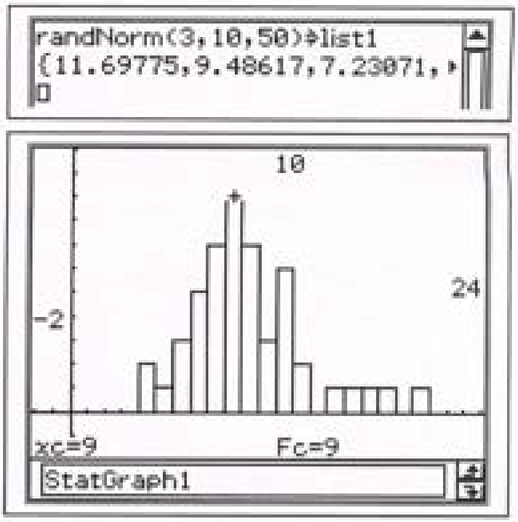
1. (a) Use your CAS calculator to generate a sample of 50 observations of the Binomial variable X with parameters $n = 10$, $p = 0.7$. Export your sample into a list.
   (b) Use the statistical graphing capability to represent this list as a histogram.
   (c) Repeat (a) and (b) to obtain a total of 10 different samples of 50 observations each.
   (d) For each sample calculate the proportion of observations that exceed 6.
   (e) Comment on your answers in (d).



2. (a) Use your CAS calculator to generate 10 samples each with 50 observations of the discrete uniform variable X, for $x = 1, 2, 3, \ldots, 10$.
   (b) Use your CAS calculator to draw the corresponding histograms.
   (c) For each sample calculate the proportion of observations that exceed 6.
   (d) Comment on your answers in (d).



3. (a) Use your CAS calculator to generate 10 samples each with 50 observations of the normal variable X with $\mu = 10$ and $\sigma = 3$.
   (b) Use your CAS calculator to draw the corresponding histograms.
   (c) For each sample calculate the proportion of observations that exceed 6.
   (d) Comment on your answers in (d).



19 Sample Proportion

19.1 Sampling distribution for sample proportion

- The table below shows the proportion of students that take the bus to school each day for ten samples of fifty students each. These samples are taken from a larger set of students called the population set or parent set.

Sample	1	2	3	4	5	6	7	8	9	10
Sample Proportion	$\frac{27}{50}$	$\frac{23}{50}$	$\frac{26}{50}$	$\frac{29}{50}$	$\frac{31}{50}$	$\frac{27}{50}$	$\frac{25}{50}$	$\frac{24}{50}$	$\frac{26}{50}$	$\frac{28}{50}$

- As discussed in the previous chapter, because of random effects, different samples will produce different values for the sample proportion. It can also be seen that the sample proportions: $\frac{27}{50}, \frac{23}{50}, \frac{26}{50}, \frac{29}{50}, \frac{31}{50}, \frac{27}{50}, \frac{25}{50}, \frac{24}{50}, \frac{26}{50}, \frac{28}{50}$ form a frequency distribution of its own. Note that this distribution is a discrete distribution (i.e. it is not possible to have $\frac{27.5}{50}$ as a sample proportion).

- The distribution formed by the sample proportion of *all* possible samples is called the *sampling distribution of the sample proportions*. There are different sampling distributions of the sample proportions for samples of different sizes.
  - The set of ten sample proportions above form a frequency distribution of sample proportions of sample size 50. As the number of samples obtained increases, the frequency distribution approaches the sampling distribution of sample proportions of sample size 50.

- Consider a population where the proportion of the population with a particular attribute is π. (For example 23% of the population is left-handed).
  - Consider a random sample of size n taken from this population.
  - Define X: No. of elements in the sample with the required attribute.
  - Define the random variable $\hat{\pi}$: proportion of sample with given attribute.
    - That is, $\hat{\pi} = \frac{X}{n}$.
  - Clearly, X is a binomial variable with parameters n and π.
    Mean for $X = n\pi$ and standard deviation for $X = \sqrt{n\pi(1-\pi)}$
  - Therefore, the mean for $\hat{\pi} = \frac{\text{Mean for X}}{n} = \frac{n\pi}{n} = \pi$

    $$\text{Standard deviation for } \hat{\pi} = \frac{\text{Standard Deviation for X}}{n} = \frac{\sqrt{n\pi(1-\pi)}}{n} = \sqrt{\frac{\pi(1-\pi)}{n}}$$
  - Hence, the random variable $\hat{\pi}$ has mean π and standard deviation $\sqrt{\frac{\pi(1-\pi)}{n}}$.

- The sampling distribution of the sample proportions of size n has the same properties as $\hat{\pi}$ the random variable representing the *sample proportion* of size n. Hence, the sampling distribution of the sample proportions of size n will have a mean of π and standard deviation $\sqrt{\dfrac{\pi(1-\pi)}{n}}$.

Example 19.1

It is known that 35% of year 12 students in a given state are aiming for university entry. Random samples of 300 students are taken and the proportion of students aiming for university entry in each sample calculated. Find the mean and standard deviation of the sampling distribution for the sample proportions of size 300.

Solution:

Mean of sampling distribution = 0.35

$$\text{Standard deviation of sampling distribution} = \sqrt{\frac{0.35\times0.65}{300}} \approx 0.02754$$

19.2 The Central Limit Theorem

- Previously, it was pointed out that the sampling distribution of sample proportions:
  - has mean = population proportion π
  - has standard deviation = $\sqrt{\dfrac{\pi(1-\pi)}{n}}$.
- The Central Limit Theorem states that the sampling distribution of sample proportions has a distribution that approaches the ***normal*** distribution as the sample size n increases.
- Hence, as the sample size n increases, the sampling distribution of sample proportions tends towards a normal distribution with:

  mean = population proportion π and standard deviation = $\sqrt{\dfrac{\pi(1-\pi)}{n}}$.
- For practical purposes, it is acceptable to treat the sampling distribution as normally distributed as long as the size of the sample $n \geq 30$.
- Note that $\hat{\pi}$ is a discrete distribution related to the binomial distribution. The Central Limit Theorem permits us to approximate $\hat{\pi}$ with a continuous normal distribution provided the sample size is large.

Example 19.2

It is known that 85% of students starting school in a particular state have been immunised against measles. Samples of 250 students starting school were taken. Describe the probability distribution for the sampling distribution of sample proportions of size 250 for students immunised against measles.

Solution:

As the size of the sample = 250 > 30, by the Central Limit Theorem,
$\hat{\pi}$ is approximately normally distributed
with mean = 0.85 and standard deviation = $\sqrt{\dfrac{0.85\times 0.15}{250}} \approx 0.02258$.

Example 19.3

It is known that 42% of students at a college have a parent who was once a student of the college. A sample of 50 students was selected from this school.

(a) Use the Central Limit Theorem to estimate the probability that the sample proportion of students with a parent who was once also a student of the college is no more than 0.5.

(b) Use the Binomial Distribution to estimate the probability that no more than 50% of students in this sample have a parent who was once a student of the college.

(c) Use an appropriate method to estimate the probability that 50% of students in this sample have a parent who was once a student of the college.

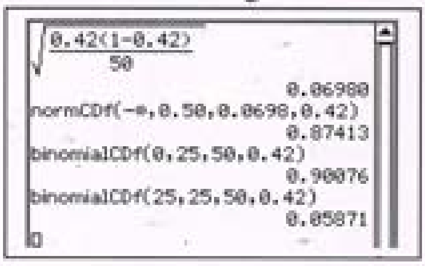
Solution:

(a) By the Central Limit Theorem, as sample size $n = 50 > 30$,
the sample proportion $\hat{\pi}$ will be approximately normally distributed with:

$$\text{mean} = 0.42 \quad \text{and} \quad \text{standard deviation} = \sqrt{\frac{0.42\times(1-0.42)}{50}} \approx 0.06980$$

Hence, $P(\hat{\pi} \le 0.50) \approx 0.87413$.

(b) Let X: No. of students with a parent who was once a student of the college out of 50.
$X \sim B(n = 50, p = 0.42)$.
Hence, $P(X \le 25) = 0.90076$.



(c) From (b), $X \sim B(n = 50, p = 0.42)$.
Hence, $P(X = 25) = 0.05871$.

Notes:

- *$P(\hat{\pi} \le 0.50)$ and $P(X \le 25)$ refer to the probability that the sample proportion is no more than 50%. As expected, the answers are slightly different. $P(\hat{\pi} \le 0.50)$ uses the normal distribution to approximate a discrete distribution without the use of a continuity correction, which in this instance is difficult to apply.*
- *Part (c) was done using the binomial variable X. To approximate $\hat{\pi}$ with a normal distribution would require the use of a continuity correction which in this case is not appropriate.*

Example 19.4

It is known that 5% of a batch of USB sticks manufactured is defective. Samples of n USB sticks are randomly selected. Find n if the standard deviation of the sample proportion of sticks that are defective is not to exceed 0.01.

Solution:

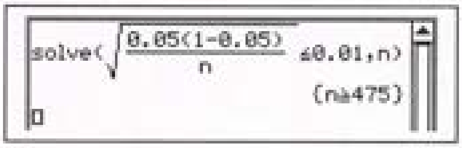
The sample proportion $\hat{\pi}$ will have mean = 0.05

and standard deviation = $\sqrt{\dfrac{0.05\times(1-0.05)}{n}}$.

For the standard deviation not to exceed 0.01:

$$\sqrt{\frac{0.05\times(1-0.05)}{n}} \leq 0.01$$

$$n \geq 475$$



Exercise19.1

1. It is known that 38% of students in a school are from Non-English Speaking Backgrounds (NESB). A sample of 80 students from this school was taken.
   (a) Describe the probability distribution for the sampling distribution of sample proportions of size 80 of students from NESB.
   (b) Use the Central Limit Theorem to estimate the probability that the sample proportion of students from NESB is at least 40%.
   (c) Use the Binomial Distribution to estimate the probability that at least 40% of students in this sample are from NESB.
   (d) Use an appropriate method to estimate the probability that 40% of students in this sample are from NESB.

2. It is known that 9% of residential homes in a city are connected to a 4G wireless network. A sample of 200 homes from this city was taken.
   (a) Describe the probability distribution for the sampling distribution of sample proportions of size 200 of residential homes connected to a 4G wireless network.
   (b) Use the Central Limit Theorem to estimate the probability that the sample proportion of residential homes connected to a 4G wireless network is at least 10% .
   (c) Use the Binomial Distribution to estimate the probability that in this sample, no more than 12% of residential homes in this city are connected to a 4G wireless network.
   (d) Use an appropriate method estimate the probability that in this sample, 10% of residential homes in this city are connected to a 4G wireless network

3. Samples of 50 observations each of a binomial variable with parameters $n = 10$ and $p = 0.47$ were taken.
   (a) Describe the probability distribution for the sample proportions $\hat{\pi}$ of size 50.
   (b) Use the Central Limit Theorem to estimate the probability that $\hat{\pi}$ lies between 0.45 and 0.48.

4. Samples of 150 observations each of a binomial variable with parameters $n = 25$ and $p = 0.63$ were taken.
   (a) Describe the probability distribution for the sample proportions $\hat{\pi}$ of sample size 150.
   (b) Use the Central Limit Theorem to estimate the probability that $\hat{\pi}$ exceeds 0.62 given that it is no more than 0.65.

5. The variable X is uniformly distributed for $0 \le x \le 24$.
   (a) Find $P(X > 20)$.
   (b) Samples of 40 observations of X were taken and the sample proportion $\hat{\pi}$ of observations exceeding 20 calculated.
      (i) Describe the probability distribution of $\hat{\pi}$.
      (ii) Use the normal approximation to $\hat{\pi}$ to estimate $P(\hat{\pi} \ge 0.2 \mid \hat{\pi} \ge 0.15)$.

6. The variable X is a discrete uniform variable for X = 1, 2, 3, 4, 5.
   (a) Find $P(X \le 2)$.
   (b) Samples of 60 observations of X were taken and the sample proportion $\hat{\pi}$ of observations not exceeding 2 calculated.
      (i) Describe the probability distribution of $\hat{\pi}$.
      (ii) Use the normal approximation to $\hat{\pi}$ to estimate $P(\hat{\pi} \le 0.3 \mid \hat{\pi} \le 0.4)$.

7. The variable X is a normal variable with mean 60 and standard deviation 15.
   (a) Find $P(X \le 80)$.
   (b) Samples of 100 observations of X were taken and the sample proportion $\hat{\pi}$ of observations not exceeding 80 calculated.
      (i) Describe the probability distribution of $\hat{\pi}$.
      (ii) Use Central Limit Theorem to estimate $P(\hat{\pi} \le 0.8 \mid \hat{\pi} \ge 0.9)$.

8. The variable X is a normal variable with mean 100 and standard deviation 20.
   (a) Find $P(X > 85)$.
   (b) Samples of 200 observations of X were taken and the sample proportion $\hat{\pi}$ of observations exceeding 85 calculated.
      (i) Describe the probability distribution of $\hat{\pi}$.
      (ii) Use Central Limit Theorem to estimate $P(\hat{\pi} \ge 0.75 \mid \hat{\pi} \le 0.8)$.

9. It is known that 22% of students in a school come from single parent homes. Samples of n students are randomly selected. Find n if the standard deviation of the sample proportion of students from single parent homes is not to exceed 0.05.

10. It is known that 19% of residents in a suburb are over the age of 65. Samples of n residents are randomly selected. Find n if the standard deviation of the sample proportion of residents over the age of 65 is not to exceed 0.1.

11. The waiting time at a pharmacy is uniformly distributed over the interval 5 to 20 minutes. In a study conducted by the pharmacist, the waiting times of several samples of 40 clients each, were recorded.
    (a) Find the mean waiting time of all clients at the pharmacy.
    (b) Describe the probability distribution that best models the distribution of the sample proportion of 40 clients with waiting times exceeding the mean waiting time.
    (c) Find the probability that a randomly chosen:
        (i) patient has to wait at least 15 minutes.
        (ii) sample has a sample proportion of clients with waiting times exceeding the mean waiting time of no more than 60%.

12. The amount of sugar dispensed by an automatic sugar dispenser is uniformly distributed over the interval 1.5 g to 2.5 g. The dispenser was used n times and the amount of sugar dispensed recorded.
    (a) For $n = 100$, describe the probability distribution that best models the distribution of the sample proportion of sample size n of sugar dispensed not exceeding 2.1 g
    (b) For $n = 100$, find the probability that a randomly chosen sample has a sample proportion of sample size n of sugar dispensed not exceeding 2.1 g of at least 61%.
    (c) Find n so that the standard deviation associated with the sampling distribution of sample proportion of sugar dispensed not exceeding 2.1 g is no more than 0.05 g.

13. A box has 1 green ball and 7 red balls. Eight balls are drawn with replacement from this box and the number of green balls noted. This procedure is repeated 36 times to form a sample of 36 observations. Define X: No of green balls drawn and let $\hat{\pi}$: the sample proportion of draws where $X \geq 3$.
    (a) Find the probability distribution for X and $\hat{\pi}$.
    (b) Use the Central Limit Theorem to find the probability that $\hat{\pi}$ has a value between 0.03 and 0.04.
    (c) Find the probability that in 20 samples of 36 such procedures; at least 4 samples would have $\hat{\pi}$ with a value between 0.03 and 0.04.

14. A box has 7 green balls and 5 red balls. Four balls are drawn without replacement from this box and the number of green balls noted. This procedure is repeated 36 times to form a sample of 36 observations. Let X: No. of green balls drawn and let $\hat{\pi}$: The proportion of draws where $X \geq 2$. .
    (a) Find the probability distribution for X and $\hat{\pi}$.
    (b) Use the Central Limit Theorem to find the probability that for any randomly chosen sample of four draws $\hat{\pi}$ has a value between 0.7 and 0.8.
    (c) Find the probability that in 50 samples of size 36 each; at least 15 samples would have $\hat{\pi}$ with a value between 0.7 and 0.8.

15. It is known that π % of teachers in a state are members of the teachers union. Samples of 144 teachers are chosen. The sampling distribution for the sample proportion of teachers of sample size 144 who are members of the union has standard deviation $\frac{\sqrt{39}}{150}$. Find π.

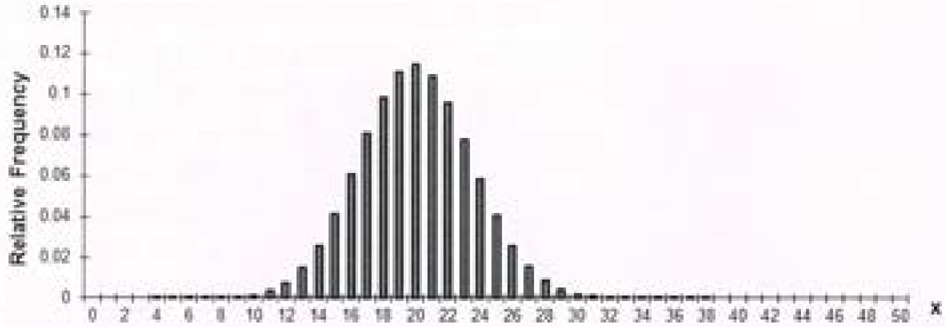
19.3 Approximating sample proportion $\hat{\pi}$ with the Normal Distribution

- As mentioned in Section 19.1, the random variable $\hat{\pi}$: the proportion of sample with a given attribute is given by $\hat{\pi} = \frac{X}{n}$ where the random variable

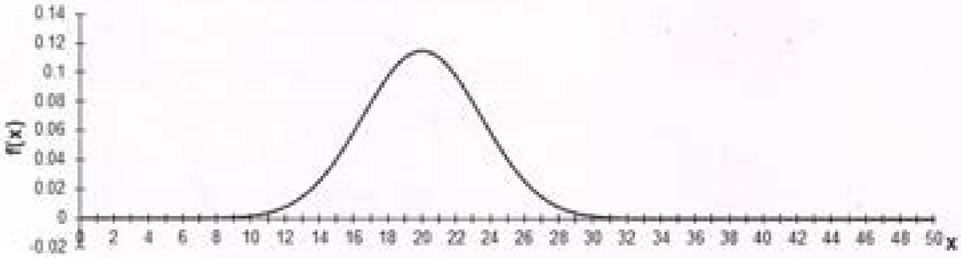
  X: No. of elements in sample with the required attribute, is a binomial variable with parameters n and π. π is the proportion of the population with the given attribute.

- In Section 19.2, it was noted that if the sample size $n \geq 30$, then by the Central Limit Theorem, $\hat{\pi}$ may be approximated by a normal variable.

- There is another justification for approximating $\hat{\pi}$ with the normal distribution. It can be shown that if X is a Binomial Variable with parameters n and p, and if $n \geq 30$ with $np > 5$ and $nq > 5$ (that is, p is neither too close to 0 or 1) then, X may approximated by a Normal Variable with mean np and variance $np(1-p)$.
  - The diagram below shows the relative frequency histogram for a binomial variable with parameters $n = 50$ and $p = 0.4$. [Which gives $np = 20$ and $np(1-p) = 12$.]



  - The diagram below shows the graph of the probability density function of a normal variable with mean 20 and variance 12.



  - Note the congruence of the two graphs.

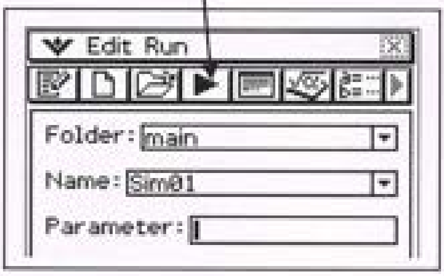
- Since, the random variable $\hat{\pi} = \frac{X}{n}$ is a discrete variable related to the binomial variable, for large n, $\hat{\pi}$ may also be approximated with a normal variable.
  - Hence, for $n \geq 30$ with $np > 5$ and $nq > 5$, $\hat{\pi} = \frac{X}{n}$ may be approximated with a normal variable with $\mu = \frac{np}{n} = p$ and $\sigma^2 = \frac{npq}{n^2} = \frac{pq}{n}$.

Hands On Task 19.1 (For Casio ClassPads Only)

- A simulated run of *b samples* each with *n* observations of a Bernoulli variable with parameter π is equivalent to a simulated run of *one* sample of *b* observations of a binomial variable with parameters *n* and π.
- In this task, we will use the first approach (use a Bernoulli variable) to simulate a frequency distribution of *b* sample proportions calculated from *b* samples each containing *n* observations of a random variable, and explore the behaviour of the frequency distribution.

We will use the program wizard to write a simple program to perform the simulation

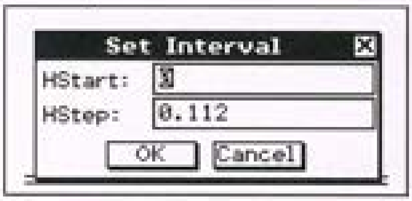
- Tap the Program Wizard.
- In the Program screen tap to start a new file (program).
- Give the new file a name, e.g. Simp01.
- Note down the parent folder, e.g. "Main".
- In the program editor, type the contents of the accompanying screen dump.
- Tap to save the program.
- Tap to exit the program editor.
- Tap to run the program.



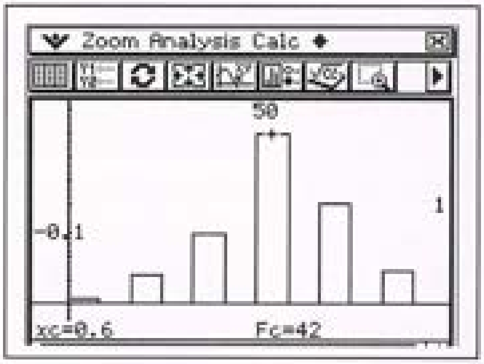
This program will:

- simulate *b* samples, each containing *n* observations of a Bernoulli variable with proportion/probability of success π.
- calculate the proportion of success in each sample.
- display the frequency distribution of sample proportions as a list
- display the frequency distribution of sample proportions in a histogram.

1. Simulate 100 samples of 5 observations of a Bernoulli variable with parameter 0.6.
    - When the program has simulated the collection of 100 samples of 5 observations and formed the frequency distribution of the sample proportions, a pop-up screen will appear.

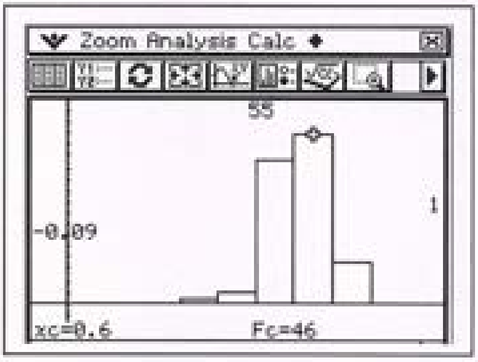


   - The screen allows you to set the drawing parameters of the frequency histogram.
        - Choose HStart: 0
          HStep: 0.1
        - Tap OK
    - The accompanying screen dump shows the final screen.



   - Tap anywhere on the graph screen.
    - You can analyse the graph by tapping the "Analysis" Menu and selecting the required tool.
    - You can also calculate the mean of the sampling distribution and its associated standard deviation using the "Calc" Menu.
    - You can have a whole lot of fun!

2. Simulate 100 samples of 50 observations of a Bernoulli variable with parameter 0.6. Be patient!



3. Use this program to observe how for large n, as the number of samples increases, the frequency distribution of the sample means $\hat{\pi}$ approaches the theoretical sampling distribution of sample proportions $N(\pi, \frac{\pi(1-\pi)}{n})$.

20 Point & Interval Estimates for π

20.1 Point Estimates for population proportion π

- Assume that the Federal Government wishes to know the proportion π of year one students in the country who have been vaccinated against measles, mumps and rubella (MMR).
- As mentioned in Chapter 18, a census of all year one students would be both time and labour intensive and costly.
- A random sample of year one students could be taken. The calculated sample proportion of year one students vaccinated against MMR, $\hat{\pi}_0$, could be used to estimate π. When $\hat{\pi}_0$ is used in this sense, it is termed a *point estimate* of the population proportion π.
- As mentioned in Chapters 18 and 19, due of random effects these point estimates $\hat{\pi}$ vary from sample to sample and form a discrete random distribution called the sampling distribution of the sample proportion. That is, when different samples (unbiased) are taken, estimates for the population proportion will vary. This is a disadvantage of using point estimates.

20.1.1 Sampling distribution for sample proportion when π is not known

- Let $\hat{\pi}_0$ be the sample proportion of a particular sample of size n.
  It can be shown that if $n \geq 30$ and for values of $(n\,\hat{\pi}_0) > 5$ and $(n\,\hat{\pi}_0)(1-\hat{\pi}_0) > 5$, the sampling distribution of sample proportions of size n will be approximately normal with mean $\hat{\pi}_0$ and standard deviation $\sqrt{\dfrac{\hat{\pi}_0(1-\hat{\pi}_0)}{n}}$.
  - In summary:
    - The population proportion π is not known.
    - Let $\hat{\pi}$ be the random variable representing the sample proportion of size n.
    - Let $\hat{\pi}_0$ be the sample proportion of *a* particular sample.
      Then *a point estimate* for π is $\hat{\pi}_0$.
      (Since, π is not known, we use $\hat{\pi}_0$ as a point estimate for π.)
    - If, the sample size n is large (≥ 30)
      and $n\,\hat{\pi}_0 > 5$ and $n\,\hat{\pi}_0(1-\hat{\pi}_0) > 5$,
      then the sampling distribution for the sample proportion of size n will be approximately normal
      with mean $\hat{\pi}_0$ and standard deviation $\sqrt{\dfrac{\hat{\pi}_0(1-\hat{\pi}_0)}{n}}$.

Example 20.1

In a random sample of 100 students, it was found that 43 of these students lived within a 5 km radius of their school.

(a) Suggest a point estimate for the true proportion of students in this state that live within a 5 km radius of their school.

(b) State the approximate probability distribution for the sampling distribution of sample proportions of students that live within 5 km of their school for samples of size 100.

(c) 200 samples of 100 students each were taken. State the approximate probability distribution for the frequency distribution of sample proportions of students that live within 5 km of their school for samples of size 100.

Solution:

(a) Sample proportion for given sample $= \dfrac{43}{100} = 0.43$.

Hence, an estimate for the true proportion of students in this state that live within a 5 km radius of their school would be 43%.

(b) As sample size $n > 30$, the sampling distribution for sample proportions of sample size 100 is approximately normal with mean 0.43 and standard deviation $\sqrt{\dfrac{0.43 \times 0.57}{100}} \approx 0.049\,508$.

(c) As the number of samples is large, the frequency distribution tends towards the sampling distribution of sample proportions of size 100.
Hence, the frequency distribution of the 200 sample proportions has an approximate normal distribution with mean 0.43 and standard deviation 0.049 508.

20.2 Probability distribution of $\dfrac{\hat{\pi} - \pi}{\sqrt{\dfrac{\hat{\pi}(1-\hat{\pi})}{n}}}$

- Consider b samples each with n observations of a random variable X. Let the proportion of the required attribute be π.
  - For each sample, the sample proportion $\hat{\pi}$ and the statistic $\dfrac{\hat{\pi} - \pi}{\sqrt{\dfrac{\hat{\pi}(1-\hat{\pi})}{n}}}$ is calculated.
  - Hence, we have a distribution for $\dfrac{\hat{\pi} - \pi}{\sqrt{\dfrac{\hat{\pi}(1-\hat{\pi})}{n}}}$.

- It can be shown that for large n ($n \geq 30$), provided $n\hat{\pi} > 5$ and $n\hat{\pi}(1-\hat{\pi}) > 5$, this distribution will be approximately standard normal.
  That is, for $n \geq 30$ and $n\hat{\pi} > 5$ and $n\hat{\pi}(1-\hat{\pi}) > 5$
  $$\frac{\hat{\pi}-\pi}{\sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}} \sim N(0, 1).$$
- As the number of samples b increases in size, the frequency distribution of $\frac{\hat{\pi}-\pi}{\sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}}$ will tend towards the sampling distribution of $\frac{\hat{\pi}-\pi}{\sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}}$.

Example 20.2

It is known that the 28% of residents in a certain state are aged 65 and above. 500 samples of 100 residents were taken. For each sample the statistic $\frac{\hat{\pi}-\pi}{\sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}}$ is calculated, where π and $\hat{\pi}$ are respectively the true proportion and sample proportion of residents aged 65 and above.

(a) Describe the sampling distribution for the sample proportion of sample size 100 of residents aged 65 and above.

(b) Describe the distribution and frequency distribution for $\frac{\hat{\pi}-\pi}{\sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}}$.

Solution:

(a) As sample size $n = 100 > 30$,
$\hat{\pi}$ is approximately normal with mean = 0.28
and standard deviation = $\sqrt{\frac{0.28 \times 0.72}{100}} = 0.0449$

(b) As $n = 100 > 30$ and as the number of samples is large, the distribution for $\frac{\hat{\pi}-\pi}{\sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}}$ and the frequency distribution of $\frac{\hat{\pi}-\pi}{\sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}}$ will be approximately standard normal.

Hands On Task 20.1 (For Casio ClassPads Only)

In this task, we will make use of a Casio ClassPad to observe how the variable $\dfrac{\hat{\pi}-\pi}{\sqrt{\dfrac{\hat{\pi}(1-\hat{\pi})}{n}}}$ tends towards a standard normal distribution.

1. Create and save a new program "Simp02" to:
   - simulate b samples, each containing n observations of a Bernoulli variable with proportion/probability of success π.
   - calculate $\hat{\pi}$ the proportion of success in each sample.
   - display the value of $\dfrac{\hat{\pi}-\pi}{\sqrt{\dfrac{\hat{\pi}(1-\hat{\pi})}{n}}}$ for each sample.
   - display the frequency histogram for $\dfrac{\hat{\pi}-\pi}{\sqrt{\dfrac{\hat{\pi}(1-\hat{\pi})}{n}}}$.

```
Simp02   N
ClrText
Input n,"Size of Sample"
Input b,"No. of Samples"
Input p,"π=? (0<π<1)","Population
Proportion π"
seq(x,x,1,b)⇒a
For 1⇒i To b
0⇒c
randBin(1,p,n)⇒x
For 1⇒j  To n
If x[j]=1
Then
c+1⇒c
IfEnd
Next
c/n⇒d
(d-p)/√(d×(1-d)/n)⇒a[i]
Print x
Next
a⇒List1
Print a
StatGraph 1,On,Histogram,List1,1
DrawStat
```

2. By observing the shape of the frequency histogram of $\dfrac{\hat{\pi}-\pi}{\sqrt{\dfrac{\hat{\pi}(1-\hat{\pi})}{n}}}$, use this program to explore how $\dfrac{\hat{\pi}-\pi}{\sqrt{\dfrac{\hat{\pi}(1-\hat{\pi})}{n}}}$ tends towards a standard normal variable for large values of n ($n \geq 30$) and values of π such that $n\pi > 5$ and $n(1-\pi) > 5$.

Exercise 20.1

1. Consider the Bernoulli variable X with probability of success π. Five samples each containing 10 observations of X were obtained and displayed as lists.

```
{0,1,1,0,0,0,1,1,1,1}
{1,1,0,1,0,0,0,0,1,1}
{1,0,0,1,1,0,1,1,1,0}
{1,1,1,1,1,1,1,1,0,0}
{1,0,0,0,1,1,0,1,1,1}
```

   (a) Calculate the proportion of successes for each sample.

   (b) Use the first sample to provide a point estimate for π and:

      (i) state the mean and standard deviation for the sampling distribution for the sample proportions of sample size 10.

      (ii) calculate the value of $\dfrac{\hat{\pi}-\pi}{\sqrt{\dfrac{\hat{\pi}(1-\hat{\pi})}{n}}}$ for each sample.

2. Consider the Bernoulli variable X with probability of success π. One sample containing 50 observations of X were obtained and displayed as shown.

```
{1,0,0,0,1,1,0,0,1,0,
1,1,1,1,1,0,0,1,0,1,
0,1,0,1,1,0,1,1,1,1,
0,0,1,1,0,0,1,0,1,0,
0,1,1,1,1,1,1,1,1,1}
```

   (a) Calculate the proportion of successes for this sample.

   (b) Use this sample to provide a point estimate for π and:

      (i) state the approximate distribution for the sampling distribution for the sample proportions of sample size 50.

      (ii) calculate the value of $\dfrac{\hat{\pi}-\pi}{\sqrt{\dfrac{\hat{\pi}(1-\hat{\pi})}{n}}}$ for this sample.

      (iii) state the probability distribution for the statistic $\dfrac{\hat{\pi}-\pi}{\sqrt{\dfrac{\hat{\pi}(1-\hat{\pi})}{n}}}$

3. Consider the binomial variable X with parameters 10 and π. Five observations were made on X and the results as shown.

```
{8,8,4,6,5}
```

   (a) State the frequency distribution of the sample proportions of successes.

   (b) Use the first observation on X as an estimate for π and:

      (i) state the mean and standard deviation for the sampling distribution for the sample proportions of sample size 10.

      (ii) calculate the value of $\dfrac{\hat{\pi}-\pi}{\sqrt{\dfrac{\hat{\pi}(1-\hat{\pi})}{n}}}$ for each observation.

4. Consider the binomial variable X with parameters 30 and π. Ten observations were made on X and the results as shown.

   {17, 10, 13, 13, 20, 14, 13, 17, 13, 18}

   (a) State the frequency distribution of the sample proportions of successes.
   (b) Use the sixth observation on X as an estimate for π and:
      (i) state the approximate probability distribution for the sampling distribution for the sample proportions of sample size 30.
      (ii) calculate the value of $\dfrac{\hat{\pi}-\pi}{\sqrt{\dfrac{\hat{\pi}(1-\hat{\pi})}{n}}}$ for each observation and hence the mean and standard deviation for the frequency distribution of $\dfrac{\hat{\pi}-\pi}{\sqrt{\dfrac{\hat{\pi}(1-\hat{\pi})}{n}}}$.

5. In a random sample of 200 students, it was found that 59 of these students were dropped off at school each morning by their parents:
   (a) Suggest a point estimate for the true proportion of students in this city that were dropped off at school by their parents.
   (b) State the approximate probability distribution for the sampling distribution of sample proportions of students in this city that were dropped off at school by their parents for samples of size 200.
   (c) 500 samples of size 200 students each were taken. State the approximate probability distribution for the frequency distribution of sample proportions of students that were dropped off at school by their parents for samples of size 200.

6. It is known that the 32% of residents in a certain city are aged below 25 years. 1 000 samples of 50 residents were taken. For each sample the statistic $\dfrac{\hat{\pi}-\pi}{\sqrt{\dfrac{\hat{\pi}(1-\hat{\pi})}{n}}}$ is calculated, where π and $\hat{\pi}$ are respectively the true proportion and sample proportion of residents aged under 25 years.
   (a) Describe the sampling distribution for the sample proportion of sample size 50 of residents aged under 25 years.
   (b) Describe the probability distribution of the frequency distribution for $\dfrac{\hat{\pi}-\pi}{\sqrt{\dfrac{\hat{\pi}(1-\hat{\pi})}{n}}}$.

20.3 Interval Estimates for population proportion π

- Interval estimates for the population proportion π involve using a point estimate and the properties of its sampling distribution to provide an interval of values for estimating π.

20.3.1 Confidence Intervals for π

- A confidence interval for π:
  - uses the sample proportion $\hat{\pi}$ as a point estimate for π,
  - provides a margin of error ($\pm e$) for the point estimate $\hat{\pi}$, thereby providing an interval of possible values for π, $\hat{\pi} - e \le \pi \le \hat{\pi} + e$,
  - and a statement of confidence in percentage terms that π would lie within the interval of values calculated.

20.3.2 Calculating Confidence Intervals for π

- Let the random variable $\hat{\pi}$: sample proportion for samples of size n, where $n \ge 30$, from a population with proportion of success π.
- As seen in the previous section, $\hat{\pi}$ may be approximated by a normal variable with mean $\mu = \pi$ and standard deviation $\sigma = \sqrt{\dfrac{\hat{\pi}(1-\hat{\pi})}{n}}$.

  Note that $\dfrac{\hat{\pi}-\pi}{\sigma} = \dfrac{\hat{\pi}-\pi}{\sqrt{\dfrac{\hat{\pi}(1-\hat{\pi})}{n}}}$ will be approximately standard normal, $Z \sim N(0, 1)$.

- Consider $P(\pi - k \le \hat{\pi} \le \pi + k) = 0.90$.

$$\Rightarrow \quad P\left(\frac{-k}{\left(\sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}\right)} \le Z \le \frac{k}{\left(\sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}\right)}\right) = 0.90.$$

Since, $\quad P(-1.645 \le Z \le 1.645) = 0.90,$

$$\frac{k}{\left(\sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}\right)} = 1.645. \qquad \Rightarrow \quad k = 1.645 \times \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}.$$

$$\Rightarrow \quad P(\pi - 1.645 \times \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}} \le \hat{\pi} \le \pi + 1.645 \times \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}) = 0.90. \qquad \text{(I)}$$

(I) can be rearranged to give:

$$P(\hat{\pi} - 1.645 \times \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}} \le \pi \le \hat{\pi} + 1.645 \times \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}) = 0.90. \qquad \text{(II)}$$

- (II) states that there is a 90% probability that π will lie in the interval
  $\hat{\pi} \pm 1.645\sqrt{\dfrac{\hat{\pi}(1-\hat{\pi})}{n}}$.
  For $\hat{\pi}_0$ as a point estimate for π, we are 90% *confident* that the interval
  $\hat{\pi}_0 \pm 1.645\sqrt{\dfrac{\hat{\pi}_0(1-\hat{\pi}_0)}{n}}$ will contain π. This interval is known as the 90% *confidence interval* for π.

- Likewise, the 95% confidence interval for π is $\hat{\pi}_0 \pm 1.960\sqrt{\dfrac{\hat{\pi}_0(1-\hat{\pi}_0)}{n}}$.
  and the 99% confidence interval for μ is $\hat{\pi}_0 \pm 2.576\sqrt{\dfrac{\hat{\pi}_0(1-\hat{\pi}_0)}{n}}$.

- Hence, if the sample size $n \geq 30$, the $100c$ % confidence interval for π is given by
  $$\hat{\pi} - z_c \times \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}} < \pi < \hat{\pi} + z_c \times \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}$$
  where $P(-z_c < Z < z_c) = c$.

- Clearly, different samples will yield varying interval estimates for π with the same confidence level $100c$ %. However, from the repeated sampling, $100c$ % of the interval estimates obtained will actually contain π.

- That is, the confidence level of $100c$ % refers to the percentage of intervals obtained through repeated sampling (of the same size) that will contain π.

- In summary, a $100c$ % confidence interval for π:
  - uses the sample proportion $\hat{\pi}$ as a point estimate for π,
  - provides a margin of error ($\pm\, z_c \times \sqrt{\dfrac{\hat{\pi}(1-\hat{\pi})}{n}}$) for the point estimate $\hat{\pi}$,

    thereby providing an interval of possible values for π, $\hat{\pi} \pm z_c \times \sqrt{\dfrac{\hat{\pi}(1-\hat{\pi})}{n}}$,
  - provides a statement of confidence in percentage terms that through repeated sampling, $100c$ % of intervals $\hat{\pi} \pm z_c \times \sqrt{\dfrac{\hat{\pi}(1-\hat{\pi})}{n}}$ obtained will contain π.

- The table below summarises the different confidence intervals for μ.

Confidence Level	Confidence interval
90%	$\hat{\pi} \pm 1.645\sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}$
95%	$\hat{\pi} \pm 1.960\sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}$
99%	$\hat{\pi} \pm 2.576\sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}$
$100c$ %	$\hat{\pi} \pm z_c \times \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}$
$100(1-\alpha)$ %	$\hat{\pi} \pm z_{\frac{\alpha}{2}} \times \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}$

- Note that $P(-z_c \leq Z \leq z_c) = c$ and $P(-z_{\frac{\alpha}{2}} \leq Z \leq z_{\frac{\alpha}{2}}) = 1-\alpha$.

- Note that the higher the level of confidence, the wider the confidence interval.

Example 20.3

It is known that in a certain city, the proportion of dwellings armed with electronic security alarm systems is π. A random sample of 150 dwellings was selected and 58 of these dwellings had such devices installed.
(a) Find a point estimate for the proportion of dwellings in this city armed with such devices.
(b) Estimate a 90% confidence interval for π.

Solution:

(a) Point estimate for π, $\hat{\pi} = \frac{58}{150} = \frac{29}{75}$.

(c) Since $n \geq 30$, by the Central Limit Theorem, $\hat{\pi} \sim N\left(\frac{29}{75}, \frac{\frac{29}{75}\left(1-\frac{29}{75}\right)}{150}\right)$.

A 90% confidence interval for π is $\frac{29}{75} \pm 1.645\sqrt{\frac{\frac{29}{75}\left(1-\frac{29}{75}\right)}{150}}$

$$\Rightarrow \quad 0.32 \leq \pi \leq 0.45.$$

Example 20.4

Let the proportion of voters in a city that support the "Ban all 4 Wheel Drive Vehicles in the City" political party be π. A random sample of 400 voters was selected and 80 indicated that they would vote for this political party. Without the use of a calculator:
(a) find a point estimate for π
(b) given $P(-2.24 \le Z \le 2.24) = 0.975$, estimate a 97.5% confidence interval for π.

Solution:

(a) Point estimate for π, $\hat{\pi} = \dfrac{80}{400} = \dfrac{1}{5}$.

(a) Since $n \ge 30$, by the Central Limit Theorem, $\hat{\pi} \sim N(\frac{1}{5}, \frac{0.2(1-0.2)}{400})$.

Since, $P(-2.24 \le Z \le 2.24) = 0.975$

$$\text{A 97.5\% confidence interval for } \pi = \frac{1}{5} \pm 2.24\sqrt{\frac{0.2(1-0.2)}{400}}$$
$$= 0.2 \pm 2.24 \times \frac{0.4}{20}$$
$$= 0.2 \pm 0.0448$$

That is, $0.15 \le \pi \le 0.24$.

Example 20.5

Let the proportion of parents/guardians in a college that support a six-day school week be π. A random sample of n parents/guardians (where $n \ge 100$) was selected and 56 indicated that they supported the proposal. Find n if the magnitude of the margin of error for the 99% confidence interval for π is 0.1.

Solution:

Point estimate for π, $\hat{\pi} = \dfrac{56}{n}$.

Since $n \ge 30$, by the Central Limit Theorem, $\hat{\pi}$ is normally distributed.

The margin of error for the 99% confidence interval is

$$e = 2.546 \times \sqrt{\frac{\frac{56}{n}\left(1-\frac{56}{n}\right)}{n}}.$$

For magnitude of margin of error = 0.1:

$$2.546 \times \sqrt{\frac{\frac{56}{n}\left(1-\frac{56}{n}\right)}{n}} = 0.1$$
$$\Rightarrow \text{integer } n = 152.$$

$$\text{solve}\left(2.546\times\sqrt{\frac{\frac{56}{n}(1-\frac{56}{n})}{n}}=0.1,n\right)$$
{n=-214.0061,n=62.8341,n=151.1720}

Example 20.6

Let the proportion of parents/guardians in a college that support a four-day school week be π. A random sample of 200 parents/guardians was selected and 78 indicated that they support the proposal. Find the level of confidence for a confidence interval for π with an error of $\pm$ 0.1.

Solution:

Point estimate for π, $\hat{\pi} = \dfrac{78}{200} = 0.39$.

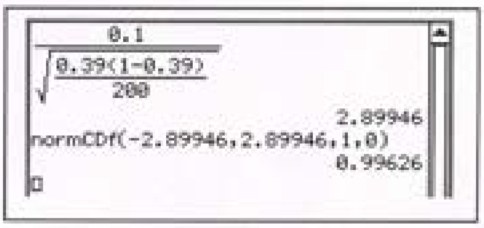
Since $n \geq 30$, by the Central Limit Theorem, $\hat{\pi}$ is normally distributed.

$$e = \pm 0.1 \quad \Rightarrow \quad z_c \times \sqrt{\frac{0.39(1-0.39)}{200}} = 0.1$$

$$z_c = 2.89946$$

But $P(-2.89946 \leq Z \leq 2.89946) = 0.99626$.

Hence, level of confidence is 99.6%.



Example 20.7

Let the proportion of people in a city that are able to "roll their tongue" be π. A sample of 400 residents in this city yielded a confidence interval for π as $0.23 \leq \pi \leq 0.29$.

(a) How many in this sample are able to "roll their tongues"?

(b) If 50 samples of 400 residents each were selected, and the associated confidence intervals for π calculated in the same manner. How many of these confidence intervals would actually contain π?

Solution:

(a) Point estimate for π, $\hat{\pi} = \dfrac{0.23 + 0.29}{2} = 0.26$.

Hence, number in sample = $0.26 \times 400 = 104$.

(b) Since $n \geq 30$, by the Central Limit Theorem, $\hat{\pi}$ is normally distributed.

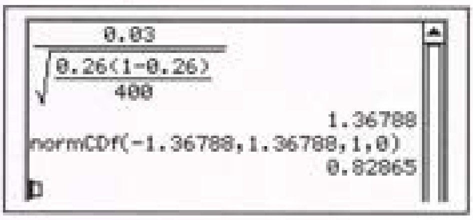
Error $e = 0.29 - 0.26 = 0.03$.

Hence, $$z_c \times \sqrt{\frac{0.26(1-0.26)}{400}} = 0.03$$

$$z_c = 1.36788$$

But $P(-1.36788 \leq Z \leq 1.36788) = 0.82865$.

Hence, level of confidence is 82.9%.

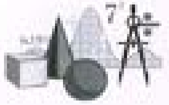


Therefore, 82.9% of the 50 confidence intervals would be expected to contain π.

That is, approximately 41 of these intervals.

20.4 Simulating Confidence Intervals for the Population proportion π

- Different samples will yield different intervals estimates for π, for the same level of confidence. For example, 500 samples will yield 500 interval estimates for π for a given confidence level. If the confidence level is 90%, then theoretically, 90% of the 500 intervals will contain the population proportion π. In this section, we will use a CAS calculator, to try to observe this phenomenon.



Hands On Task 20.2 **(For Casio ClassPads Only)**

In this task, we will use a Casio ClassPad to simulate the construction of confidence intervals for the population proportion π based on a binomial variable with parameters n and π.

1. Create and save a new program "Simp03".
   This program will:
   - simulate b samples each containing n observations of a Bernoulli variable with proportion of success π.
   - construct a $100c$ % confidence interval for π for each sample and display them on the screen.

```
Simp03    N
ClrText
Input n,"Size of Sample"
Input b,"No. of Samples"
Input p,"π=? (0<π<1)","Population Proportion π"
Input c,"Level 0<c<1","Confidence"
invNormCDf("C",c,1,0)⇒d
Print "π="
Locate 20,1,p
randBin(n,p,b)⇒x
For 1⇒i To b
x[i]/n⇒a
[a+d×(√(a×(1-a)/n)),a-d×(√(a×(1-a)/n))]⇒y
Print y
Next
```

- The accompanying screen dump shows a simulation of 10 samples of 100 observations each of a Bernoulli variable with proportion of success 0.5. Ten 90% confidence intervals are displayed. In this simulation, nine of the intervals actually contain the true proportion $\pi = 0.5$.
  In simulations, this need not necessarily be true all the time.

```
π= 0.5
[[0.4077737688,0.5722262312]]
[[0.4378231391,0.6021768609]]
[[0.4277737688,0.5922262312]]
[[0.3388168499,0.5011831501]]
[[0.4277737688,0.5922262312]]
[[0.4681695653,0.6318304347]]
[[0.4077737688,0.5722262312]]
[[0.3681695653,0.5318304347]]
[[0.3780209177,0.5419790823]]
[[0.3291006204,0.4908993796]]
```

2. Explore the phenomenon that $100c$% of the confidence intervals will actually contain the population proportion π.

Exercise 20.2

1. To estimate π the proportion of year 12 students in a school with mass above 65 kg, a random sample of 30 students were selected. The mass (kg) of these students were:
 63.4, 54.6, 56.5, 69.2, 59.4, 61.5, 45.6, 66.7, 49.8, 57.3, 64.3, 66.9, 74.2, 61.4, 62.5
 62.4, 63.1, 64.2, 58,7, 56.1, 55.3, 59.4, 62.4, 67.3, 57.8, 45.5, 48.7, 69.2, 51.9, 57.4.
 Use this sample to find a point estimate for π.

2. To estimate the daily travelling times to school, 60 students were surveyed. The results are shown in the accompanying table.
 (a) Use this sample to obtain a point estimate for π, the proportion of students with travelling times exceeding 15 minutes.
 (b) State the probability distribution for the sample proportion of students with travelling times exceeding 15 minutes.

Travelling Time (mins)	No. of Students
$0 < t \le 5$	5
$5 < t \le 10$	20
$10 < t \le 15$	15
$15 < t \le 20$	10
$20 < t \le 25$	5
$25 < t \le 30$	3
$30 < t \le 60$	2

3. The proportion of dwellings in a certain suburb equipped with roof-top solar panels connected to the main electricity supply grid is π. A random sample of 200 dwellings was selected and 42 of these dwellings had such devices installed.
 (a) Find a point estimate for π.
 (b) Estimate a 95% confidence interval for π.
 (c) Find the size of the next sample of dwellings if the error margin for a 95% confidence interval for π is no more than 0.1.

4. The proportion of dwellings in a certain suburb equipped with an underground bore for garden reticulation is π. A random sample of 120 dwellings was selected and 68 of these dwellings were equipped with underground bores.
 (a) Find a point estimate for π.
 (b) Estimate a 90% confidence interval for π.
 (c) Find the size of the next sample of dwellings if we wish to be 90% confident that the point estimate for π is not to differ from the true value of π by more than 0.08.

5. The proportion of voters in a state that support the ban of live animal exports is π. A sample of 400 urban voters was selected and 320 indicated that they support the ban.
 (a) Without the use of a calculator find a point estimate for π.
 (b) Determine with reasons if the point estimate in (a) would be a fair estimate for π.
 (c) Given $P(-2 \le Z \le 2) = 0.955$, without the use of a calculator, estimate a 95.5% confidence interval for π.

6. The proportion of mathematics teachers in a state that believe that CAS calculators should be banned in examinations is π. A sample of 100 mathematics teachers was selected and 40 were found to agree with the ban of CAS calculators in examinations.
 (a) Without the use of a calculator find a point estimate for π.
 (b) Given $P(-1.5 \le Z \le 1.5) = 0.866$, without the use of a calculator, estimate a 86.6% confidence interval for π.

7. The proportion of adults in a state that self-diagnose their health condition through the internet is π. In a random sample of n adults (where $n \geq 100$) 88 adults indicated that they self-diagnose their health condition through the internet. Find n if the magnitude of the margin of error for the 95% confidence interval for π is 0.05.

8. The proportion of students in a state that have Facebook accounts is π. A random sample of n students (where $n \geq 100$) was selected and 50 indicated that they *do not* have Facebook accounts. Find n if the magnitude of the margin of error for the 99% confidence interval for π is 0.05.

9. Let the proportion of parents/guardians in a high school that support the proposal that students be required to spend one Saturday morning each term to clean the school toilets is π (in some East Asian countries, students are required to turn up each Saturday morning during term time to clean the school toilets). A random sample of 200 parents/guardians was selected and 160 indicated that they support the proposal. Find the level of confidence for a confidence interval for π with an error of $\pm$ 0.05.

10. Let the proportion of students in a high school that support the proposal that parents/guardians be required to spend one Saturday morning each term to help maintain the school grounds and lawn is π A random sample of 100 students was selected and 90 students indicated that they support the proposal. Find the level of confidence for a confidence interval for π with an error of $\pm$ 0.08.

11. A supermarket chain intends opening their stores in a city, 24 hours, seven days a week. To estimate π the proportion of people in a city who would actually make use of this 24 hour service, a sample of 1000 residents in this city yielded a confidence interval for π as $0.28 \leq \pi \leq 0.32$.
    (a) How many in this sample indicated that they would make use of this 24 hour service?
    (b) If 20 samples of 1000 residents each were selected, and the associated confidence intervals for π calculated in the same manner. How many of these confidence intervals would be expected to contain π?

12. To estimate the true proportion π of the residents of an Australian state that would support Australia becoming a republic, a sample of 1 000 residents were interviewed. 650 residents in this sample indicated that Australia should become a republic.
    (a) Calculate approximate 90%, 95% and 99% confidence intervals for π.
    (b) In a second sample of 1 000 residents from another part of the same state, 490 residents supported the idea of Australia becoming a republic. Determine with reasons if the residents of the second sample were less in favour of Australia becoming a republic.

13. 500 students attended a University lecture. 100 of these students were tested immediately after the lecture. 61 of these students passed the test taken.
    (a) Calculate approximate 90%, 95% and 99% confidence intervals for π the proportion of students who will pass this test.
    (b) A second group of 100 students who attended the same lecture were told to take a 30 minute nap after which they were given the same test. 78 of these students passed the given test. Determine with reasons if the performance of the second group was better than that of the first group and if so, determine if that could be attributed to the nap the students had.

20.5 Level of significance **Extension**

- Let the random variable $\hat{\pi}$: sample proportion for a population proportion π from a sample of size n, where $n \geq 30$.

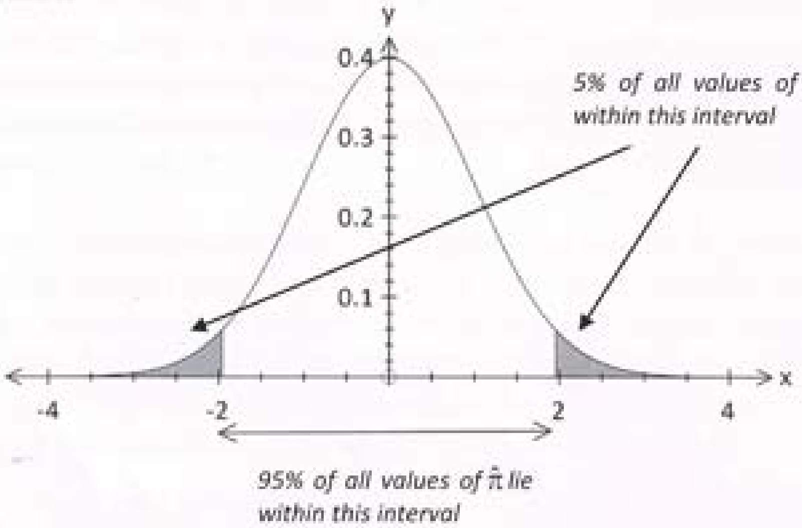
- By the Central Limit Theorem, $\hat{\pi}$ may be approximated by a normal variable with mean $\mu = \pi$ and standard deviation $\sigma = \sqrt{\dfrac{\pi(1-\pi)}{n}}$.

- Then 95% of all values of $\hat{\pi}$ would lie in the interval $\pi \pm 1.960 \times \sqrt{\dfrac{\pi(1-\pi)}{n}}$.

  That is, 5% of all values of $\hat{\pi}$ would lie outside the interval $\pi \pm 1.960 \times \sqrt{\dfrac{\pi(1-\pi)}{n}}$.

- Let $\hat{p}$ be the sample proportion for one of the samples taken.
  - If $\hat{p}$ is *outside* the interval $\pi \pm 1.960 \times \sqrt{\dfrac{\pi(1-\pi)}{n}}$,
    then it is concluded that the sample proportion $\hat{p}$ *is significantly different* from the population proportion π at the 5% level of significance.
  - If $\hat{p}$ is *inside* the interval $\pi \pm 1.960 \times \sqrt{\dfrac{\pi(1-\pi)}{n}}$,
    then it is concluded that the sample proportion $\hat{p}$ *is not significantly different* from the population proportion π at the 5% level of significance.

- This is represented visually in the diagram below.
  The shaded region is called the *critical region*.
  If $\hat{p}$ falls inside this region, then it is significantly different from π at the 5% level of significance.



Example 20.8

"Ten10" laptop batteries are manufactured with the design feature that 90% of these batteries when disconnected from mains power supply under "normal use" will continue to power the laptops for at least 10 hours and 10 minutes. A random sample of 36 such batteries yielded 32 batteries which when disconnected from the mains power supply continued to power the laptops for at least 10 hours and 10 minutes. Determine with reasons if the sample proportion is significantly different at a 10% level.

Solution:

Let $\hat{\pi}$: proportion of batteries that when disconnected from mains power supply will continue to power the laptops for at least 10 hours and 10 minutes.

Since $n \geq 30$, by the Central Limit Theorem, $\hat{\pi} \sim N(0.90, \frac{0.90(1-0.90)}{400})$.

10% of all values of $\hat{\pi}$ lie outside the interval $0.90 \pm 1.645 \times \sqrt{\frac{0.90(1-0.90)}{400}}$.

That is, 10% of all values of $\hat{\pi}$ lie outside the interval $0.87533 \leq \pi \leq 0.92468$

From sample collected, sample proportion, $\hat{p} = \frac{32}{36} = 0.88889$.

Hence $\hat{p}$ does not lie within the 10% critical region.

Therefore, sample proportion is not significantly different at the 10% level.

Example 20.9

It is known that 3% of the residents of a country are allergic to a certain anti-inflammatory drug. A random sample of 3000 patients found that 105 patients were allergic to this drug. Determine the level of significance for the proportion of those in this sample who are allergic to this drug.

Solution:

Let $\hat{\pi}$: proportion of those allergic to the drug.

Since $n \geq 30$, by the Central Limit Theorem, $\hat{\pi} \sim N(0.03, \dfrac{0.03(1-0.03)}{3000})$.

Sample proportion $\hat{p} = \dfrac{105}{3000} = 0.035$

The difference between the sample proportion $\hat{p}$ and the true proportion $\pi = 0.035 - 0.03$

Hence, error = 0.005

Therefore, $$z_c \times \sqrt{\frac{0.03(1-0.03)}{3000}} = 0.005$$

$$\Rightarrow z_c = 1.60540.$$

But $P(-1.60540 \leq Z \leq 1.60540) = 0.89159$.

Hence, the sample proportion and the true proportion is significantly different at the $(1 - 0.89159) \times 100 = 10.8\%$ level.

Exercise 20.3

1. It is known that 34% of students in a city start school each day without adequate breakfast. A random sample of 200 students found that 60 of these students had no breakfast on the day the survey was conducted.
   Determine with reasons if the sample proportion is significantly different at a:
   (a) 10% level (b) 5% level (c) 1% level.

2. It is known that 15% of male adults in a certain city suffer from sleep apnoea. A random sample of 300 male adults revealed that 60 suffered from sleep apnoea.
   Determine with reasons if this sample proportion is significantly different at a:
   (a) 10% level (b) 5% level (c) 1% level.

3. In a large apartment complex, 72% of the apartments are occupied by owner occupiers. In a random sample of 50 apartments in this complex, 30 were occupied by owner occupiers. Determine with reasons if this sample proportion is significantly different at a:
   (a) 8% level (b) 2% level

4. It is known that 8% of packets of sugar packed by a machine are "underweight". A random sample of 200 packets revealed that 21 of these packets were "underweight".
   (a) Determine with reasons if the proportion of "underweight" packets in this sample is significantly different at the 5% level.
   (b) Determine the level of significance for the proportion of "underweight" packets in this sample.

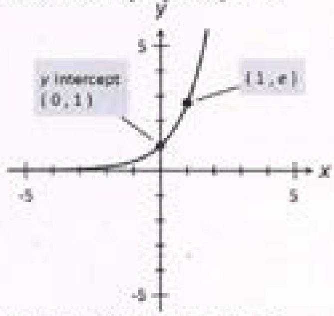
5. It is known that teaching is the first profession of 75% of mathematics teachers in a certain country. A sample of n mathematics teachers found that 78% chose teaching as their first profession.
   (a) For $n = 500$, determine the level of significance for the proportion of mathematics teachers with teaching as their first profession in this sample.
   (b) Find n if this sample proportion is to be significant at the 5% level.

6. It is known that in a certain state, 12% of all year 12 students offered places at a university take a gap year before starting university. In a sample of 1200 year 12 students offered places at a university, 160 intended to take a gap year before starting university.
   (a) Determine the level of significance for the sample proportion for this sample.
   (b) How large should the next sample be if the sample proportion of the next sample is to be significant at the 1% level? Assume that the value of the sample proportion in the next sample has the same value as that in the first sample.

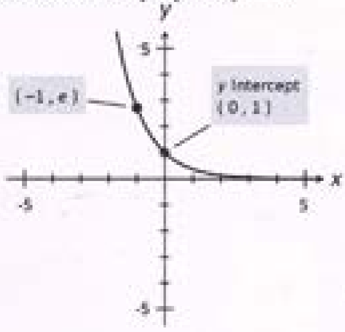
Answers

Exercise 1.1

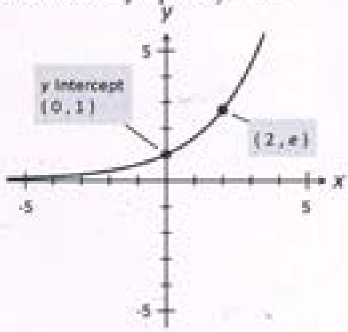
1. (a) −1 (exact) (b) 1.3863
   (c) −40.2359 (d) 1.9188
   (e) 6.9315 (f) 17.3287
2. (a) Horizontal Asymptote $y = 0$.



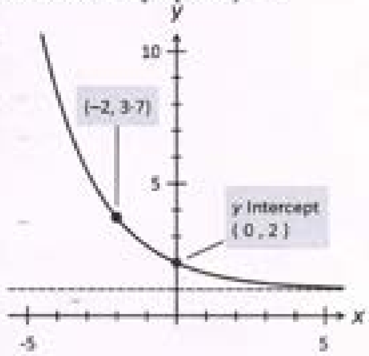
2. (b) Horizontal Asymptote $y = 0$.



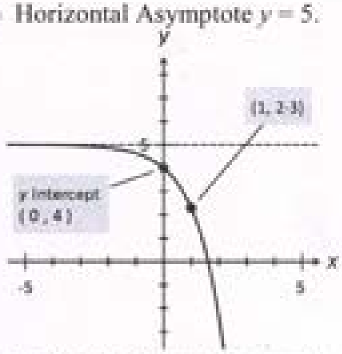
(c) Horizontal Asymptote $y = 0$.



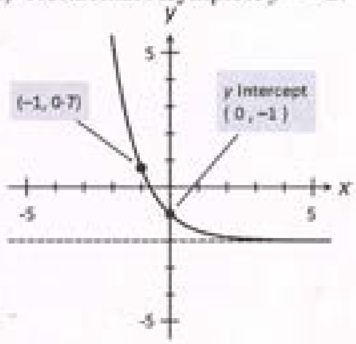
(d) Horizontal Asymptote $y = 1$



(e) Horizontal Asymptote $y = 5$.



(f) Horizontal Asymptote $y = -2$.



Exercise 1.2

1. (a) 20 million (b) 1.89% p.a.
   (c) 32.1 million (d) 21.5 years
   (e) 36.7 years
2. (a) $P = 650e^{0.09t}$, $t \geq 0$
   (b) 932 (c) $t = 4.8$ years
   (d) $t = 15.4$ years
3. (a) 4 000 (b) decay rate 0.1% p.a.
   (c) 3 980 (d) $t = 693.1$ years
4. (a) $A = 50e^{-0.01t}$ $t \geq 0$
   (b) 18.4 g (c) 19.7 g (d) 391.2 yrs
5. (a) 4.53 million
   (b) $t > 160.8$ years or in the year 2141
   (c) Highly improbable for this to occur. PNG is too poor at this stage and does not have the resources to sustain such a large population.
6. (a) $t > 123.11$ years, or in the year 2104/2105.
   (b) India 1693 million (China 1464 million, Pakistan 272 million)
7. (a) Substance X
   (b) $t = 18.3$ years or in the year 2018 (2019)
   (c) $t = 42.2$ years or in the year 2042 (2043)
8. (a) $t = 22.5$ months
   (b) As the population of B declines, there will be an insufficient supply of food for A. As such, the population of A cannot grow indefinitely. Hence, the model is valid only for a restricted period of time.
9. (a) 9.3% p.a. (b) 2.3 m (c) 64.5 m
   The difference in heights is significantly more than double.. Doubling the growth rate gave rise to a height that is 2.54 times larger than the previous height.
   (d) 2.54 m/year

10. (a) 64.9% p.a. (b) 1.67 tonnes
    (c) 1.17 tonnes/year
11. (a) Initial drug concentration = 2.00 mg/mL
    continuous growth rate = −9.04% per hour
    (b) $t = 15.3$ hours
12. (a) Initial number = 4 495
    continuous growth rate = 8.51% per month
    (b) 8 290
13. 60.7 g 14. 195.6 g
15. 2321.9 hours ago. 16. 2.65 g

Exercise 2.1

1. (a) $6(x+1)^5$ (b) $-10(1-x)^9$
   (c) $8(2x+1)^3$ (d) $-4(1-x/2)^7$
   (e) $\frac{1}{2}(1+x/3)^{1/2}$ (f) $\frac{1}{2}(x/5-1)^{3/2}$
   (g) $16x(1+2x^2)^3$ (h) $15x^2(x^3-1)^5$
   (i) $(-25/2)x^4(5-x^5)^{3/2}$ (j) $x(x^2-4)^{-1/2}$
2. (a) $24x(1+3x^2)^3$ (b) $3x(x^2/2-2)^2$
   (c) $3x(6+x^2)^{1/2}$ (d) $5x^2(3+x^3)^{2/3}$
   (e) $10x^{-3}(1-x^{-2})^4$ (f) $-12x^{-4}(x^{-3}+1)^3$
   (g) $2x^{-1/2}(1+x^{1/2})^3$ (h) $x^{-2/3}(2+x^{1/3})^2$
   (i) $(\frac{2}{3})x^{-2/3}(x^{1/3}-1)^{1/2}$
3. (a) $3(1+2x)(1+x+x^2)^2$
   (b) $4(3x^2-2)(1-2x+x^3)^3$
   (c) $3(-1/x^2+1)(1+1/x+x)^2$
   (d) $4(x^{-1/2}/2+1)(1+x^{1/2}+x)^3$
4. (a) $-1/(1+x)^2$ (b) $2/(1-x)^3$
   (c) $-2x/(2+x^2)^2$ (d) $8x/(1-x^2)^3$
5. (a) $2x^{-1/2}(1-x^{1/2})^{-2}$ (b) $(-3/2)(1+x)^{-3/2}$
   (c) $x^{-1/2}(1+x^{1/2})^{-3}$
   (d) $-(1+2x)/(1+x+x^2)^2$
6. (a) $t(1+t^2)^{-1/2}$ (b) $-t(1+t^2)^{-3/2}$
   (c) $2t-1/(1+t)^2$
   (d) $-3(1+t)^{-2}[1+1/(1+t)]^2$
7. $f'(t)=4t(1+t^2)$, $f'(1)=8$
8. $f'(t)=2t/(16-t^2)^2$, $f'(1)=2/225$
9. $f'(t)=-24t^2(1-2t^3)^3$, $t=0, 0.7937$
10. $f'(t)=(t/4-1)^3$, $t=4$

Exercise 2.2

1. (a) $6x-5$ (b) $-4x-1$
   (c) $-x/3+5/3$
   (d) $(1+2x)(4-x^2)-2x(1+x+x^2)$
   (e) $\frac{1}{2}x^{-1/2}(1-x^2)-2x(1+x^{1/2})$
   (f) $(-1/x^2)(1+x-x^2)+(1+1/x)(1-2x)$
   (g) $(-1/2)x^{-3/2}(x^3+1)+3x^2(2+x^{-1/2})$
   (h) $(1-1/x^2)(x^2-4)+2x(x+1/x)$
2. (a) $2(x+1)(x+2)+(x+1)^2$
   or $(x+1)(3x+5)$
   (b) $-(x+3)^3+3(1-x)(x+3)^2$
   or $-4x(x+3)^2$
   (c) $(2x+1)^4+6(x^2+x+1)(2x+1)^2$
   or $(2x+1)^2(10x^2+10x+7)$
   (d) $(2x+1)(1-x-x^2)^3$
   $-3(1+2x)(x^2+x+1)(1-x-x^2)^2$
   or $-2(2x+1)(2x^2+2x+1)(1-x-x^2)^2$
3. (a) $3(x+2)(2x+1)^{1/2}+(2x+1)^{3/2}$
   or $(2x+1)^{1/2}(5x+7)$
   (b) $-(4x+3)^{3/2}+6(1-x)(4x+3)^{1/2}$
   or $(4x+3)^{1/2}(-10x+3)$
   (c) $(x-1)(1+2x)^{-1/2}+(1+2x)^{1/2}$
   or $3x(1+2x)^{-1/2}$
   (d) $3(3-2x)^{1/2}-(1+3x)(3-2x)^{-1/2}$
   or $(3-2x)^{-1/2}(8-9x)$
4. (a) $2(t+1)(t-2)^3+3(t+1)^2(t-2)^2$
   or $(t+1)(t-2)^2(5t-1)$
   (b) $4(2t+1)(t+2)^2+2(2t+1)^2(t+2)$
   or $2(2t+1)(t+2)(4t+5)$
   (c) $-(1-t/2)(1+2t)^3+6(1-t/2)^2(1+2t)^2$
   or $5(1-t/2)(1+2t)^2(1-t)$
   (d) $3(1+2t)^{1/2}(1-t)^2-2(1+2t)^{3/2}(1-t)$
   or $(1+2t)^{1/2}(1-t)(1-7t)$
   (e) $6(1+4t)^{1/2}(3+2t)^{5/2}$
   $+5(1+4t)^{3/2}(3+2t)^{3/2}$
   or $(1+4t)^{1/2}(3+2t)^{3/2}(23+32t)$
   (f) $(1+3t)^{-2/3}(1+2t)^{3/2}$
   $+3(1+3t)^{1/3}(1+2t)^{1/2}$
   or $(1+3t)^{-2/3}(1+2t)^{1/2}(4+11t)$
   (g) $4(1+3t)^{1/3}(1+2t)^{1/2}$
   $+(1+3t)^{4/3}(1+2t)^{-1/2}$
   or $(1+3t)^{1/3}(1+2t)^{-1/2}(5+11t)$
   (h) $(1+5t)^{-4/5}(1+6t)^{1/3}$
   $+2(1+5t)^{1/5}(1+6t)^{-2/3}$
   or $(1+5t)^{-4/5}(1+6t)^{-2/3}(3+16t)$
5. (a) $4t(t^2+1)(1-t^3)^2-6t^2(t^2+1)^2(1-t^3)$
   or $2t(t^2+1)(1-t^3)(2-3t-5t^3)$

5. (b) $6t(t^2-1)^2(1-t^4/2)^2 - 4t^3(t^2-1)^3(1-t^4/2)$
   or $2t(t^2-1)^2(1-t^4/2)(3+2t^2-7t^4/2)$
6. (a) $(t+2)^{-1} - (t+1)(t+2)^{-2}$ or $(t+2)^{-2}$
   (b) $(1-t)^{-1} + (t+3)(1-t)^{-2}$ or $4(1-t)^{-2}$
   (c) $2(t+1)(2+t)^{-1} - (t+1)^2(2+t)^{-2}$
   or $(t+1)(2+t)^{-2}(3-t)$
   (d) $(1+t)^{-2} - 2(t-1)(1+t)^{-3}$
   or $(1+t)^{-3}(3-t)$
7. (a) $(x-1)^{-1/2} - ½(x+1)(x-1)^{-3/2}$
   or $½(x-1)^{-3/2}(x-3)$
   (b) $-3x(x^2+1)^{-5/2}(x-1) + (x^2+1)^{-3/2}$
   or $(x^2+1)^{-5/2}(-2x^2+3x+1)$
   (c) $2t(1+t)^{-1/2} - ½t^2(1+t)^{-3/2}$
   or $½t(1+t)^{-3/2}(3t+4)$
   (d) $2(1-t^2)^{-3/2} + 6t^2(1-t^2)^{-5/2}$
   or $2(1-t^2)^{-5/2}(1+5t^2)$
8. $f'(x) = 2(x+1)(1-x) + (x+1)^2$, $f'(0) = 1$
9. $f'(x) = -2x(1+2x)^3 + 6(1-x^2)(1+2x)^2$
   $f'(1) = -54$
10. $v'(t) = (t-1)(3t+1)$, $t = -1/3, 1$
11. $P'(t) = -(1-t/2)(1+2t^2) + 4t(1-t/2)^2$
   $t = 1/2, 2$
12. $s'(t) = 3(1-t^2)(1+2t)^{1/2} - 2t(1+2t)^{3/2}$
   $t = -1/2, (\sqrt{22}-1)/7$

Exercise 2.3

1. (a) $-2/x^3$ (b) $-3/(4x^4)$
   (c) $-1/(x+1)^2$ (d) $2/(x+1)^3$
   (e) $-2x/(x^2+1)^2$ (f) $-3/[2(x+1)^{5/2}]$
   (g) $1/[2(x-1)^{3/2}]$ (h) $3/[2(1-x)^{5/2}]$
2. (a) $1/(1-x)^2$ (b) $-2/(1+3x)^2$
   (c) $7/(1-6x)^2$ (d) $5/(1-3x)^2$
   (e) $(2x+x^2)/(1+x)^2$ (f) $(-2-2x^2)/(1-x^2)^2$
   (g) $-2x/(1+x^2)^2$ (h) $2x/(1+x^2)^2$
3. (a) $([(1+x)^2 - 2x(1+x)]/(1+x)^4$
   or $(1-x)/(1+x)^3$
   (b) $[-2(1-x)^2 - 4x(1-x)]/(1-x)^4$
   or $-2(1+x)/(1-x)^3$
   (c) $[2x(1-2x)^2 + 4x^2(1-2x)]/(1-2x)^4$
   or $2x/(1-2x)^3$
   (d) $[-12x^2(2+x)^2 + 8x^3(2+x)]/(2+x)^4$
   or $-4x^2(6+x)/(2+x)^3$
3. (e) $[2(1+x^2)^2 - 8x^2(1+x^2)]/(1+x^2)^4$
   or $2(1-3x^2)/(1+x^2)^3$
   (f) $[-2x(1-x^2)^2 - 4x^3(1-x^2)]/(1-x^2)^4$
   or $-2x(1+x^2)/(1-x^2)^3$
   (g) $[(1-x)^2 + 2(1+x)(1-x)]/(1-x)^4$
   or $(3+x)/(1-x)^3$
   (h) $[2(1-x)(1+x) + (1+x)^2]/(1-x)^2$
   or $(1+x)(3-x)/(1-x)^2$
   (i) $[-2(4-x)(1+2x)^2$
   $- 4(1+2x)(4-x)^2]/(1+2x)^4$
   or $-18(4-x)/(1+2x)^3$
   (j) $[4(3+2x)(4-5x)^2$
   $+ 10(4-5x)(3+2x)^2]/(4-5x)^4$
   or $46(3+2x)/(4-5x)^3$
   (k) $[2x(1-3x)^2 + 6(1-3x)(1+x^2)]/(1-3x)^4$
   or $2(x+3)/(1-3x)^4$
   (l) $[-4x(1-x^2)(1+x^2)$
   $- 4x(1+x^2)(1-x^2)^2]/(1+x^2)^4$
   or $-8x(1-x^2)/(1+x^2)^3$
4. (a) $-6(4t-3)^{-5/2}$ (b) $2(1+2t)^{-3/2}$
   (c) $2t^{-1/2}(1-t^{1/2})^{-2}$
   (d) $-9t^{-1/2}(1+t^{1/2})^{-5/2}/2$
   (e) $[2(1+t)^{3/2} - 3t(1+t)^{1/2}]/(1+t)^3$
   or $(2-t)/(1+t)^{5/2}$
   (f) $[-4(1-2t)^{1/2} - 4t(1-2t)^{-1/2}]/(1-2t)$
   or $-4(1-t)/(1-2t)^{3/2}$
   (g) $[3t(1+2t)^{1/2} - (1+2t)^{3/2}]/t^2$
   or $(1+2t)^{1/2}(t-1)/t^2$
   (h) $[-4t(3-4t)^{-1/2} - 2(3-4t)^{1/2}]/4t^2$
   or $(3-4t)^{-1/2}(2t-3)/(2t^2)$
5. (a) $[(1-x)^{3/2} + (3/2)(1+x)(1-x)^{1/2}]/(1-x)^3$
   or $(5+x)/[2(1-x)^{5/2}]$
   (b) $[-(1+2x)^{1/2} - (1-x)(1+2x)^{-1/2}]/(1+2x)$
   or $-(2+x)/(1+2x)^{3/2}$
   (c) $[3x(1+2x)^{1/2} - (1+2x)^{3/2}]/x^2$
   or $(1+2x)^{1/2}(x-1)/x^2$
   (d) $[-2(1+2x)(3-4x)^{-1/2} - 2(3-4x)^{1/2}]/(1+2x)^2$
   or $-4(3-4x)^{-1/2}(2-x)/(1+2x)^2$
   (e) $[-2(1+x^2)^{3/2} + 6x^2(1+x^2)^{1/2}]/(1+x^2)^3$
   or $-2(1-2x^2)/(1+x^2)^{5/2}$
   (f) $[2x(1-x^2)^{1/2} + x^3(1-x^2)^{-1/2}]/(1-x^2)^{3/2}$
   or $x(2-x^2)/(1-x^2)^{3/2}$

5. (g) $[(1-x^2)^{3/2} + 3x(1+x)(1-x^2)^{1/2}]/(1-x^2)^3$
   or $(2x^2 + 3x + 1)/(1-x^2)^{5/2}$
   (h) $[-x(1+x^2)(1-x^2)^{-1/2}$
   $- 2x(1-x^2)^{1/2}]/(1+x^2)^2$
   or $-x(1-x^2)^{-1/2}(3-x^2)/(1+x^2)^2$
6. (a) $[½(1-\theta)^{1/2}(1+\theta)^{-1/2}$
   $+ ½(1+\theta)^{1/2}(1-\theta)^{-1/2}]/(1-\theta)$
   or $1/[(1-\theta)^{3/2}(1+\theta)^{1/2}]$
   (b) $[-\theta(1+\theta^2)^{1/2}(1-\theta^2)^{-1/2}$
   $- \theta(1-\theta^2)^{1/2}(1+\theta^2)^{-1/2}]/(1+\theta^2)$
   or $-2\theta/[(1+\theta^2)^{3/2}(1-\theta^2)^{1/2}]$
7. $f'(x) = (1-x^2)/(1+x^2)^2$
   (a) 1 (b) ±1
8. $f'(x) = (9x^2 + 18x + 4)/(4-9x^2)^2$
   (a) 1/4 (b) $-1 \pm (\sqrt{5})/3$
9. $v'(t) = [2t(1+2t)^2 - 4t^2(1+2t)]/(1+2t)^4$
   (a) 0 (b) 0
10. $v'(t) = [(1+t)^{3/2} - (3/2)t(1+t)^{1/2}]/(1+t)^3$
   (a) 1 (b) 2

Exercise 2.4

1. (a) $12x$ (b) $-½(x^{-3})$
   (c) $3(x+1)^2$ (d) $-¼(x+1)^{-3/2}$
2. (a) $(1+x)^3 + 3x(1+x)^2$
   or $(1+x)^2(1+4x)$
   (b) $4x^3(1+x^{1/2})^2 + x^{7/2}(1+x^{1/2})$
   or $(1+x^{1/2})(4x^3 + 5x^{7/2})$
   (c) $(3/2)(1+x)^{1/2}(1+3x) + 3(1+x)^{3/2}$
   or $(3/2)(1+x)^{1/2}(3+5x)$
   (d) $-3x(1-x^2)^{1/2}(1+4x) + 4(1-x^2)^{3/2}$
   or $(1-x^2)^{1/2}(-16x^2 - 3x + 4)$
3. (a) $y' = 8(1+2x)^3$, $y'' = 48(1+2x)^2$
   (b) $y' = (-45/2)(2-15x)^{1/2}$
   $y'' = (675/4)(2-15x)^{-1/2}$
   (c) $y' = -9(1-6x)^{1/2}$, $y'' = 27(1-6x)^{-1/2}$
   (d) $y' = 2(1-4x)^{-3/2}$, $y'' = 12(1-4x)^{-5/2}$
4. (a) $v' = 4t + 4t^3$, $v'' = 4 + 12t^2$
   (b) $v' = -6t(1-t^2)^2$,
   $v'' = -6(1-t^2)^2 + 24t^2(1-t^2)$
5. (a) $720x^7$, $30\,240x^5$
   (b) $(-15/8)x^{-7/2}$, $(-945/32)x^{-11/2}$
6. $f''(x) = 48(5+2x)^2$
   (a) 1 200 (b) −5/2

7. (a) $x = 1$, $f''(1) = 0$
   (b) $x = \pm 2$, $f''(-2) = -12$, $f''(2) = 12$
   (c) $x = 1$ or 2, $f''(1) = -6$, $f''(2) = 6$

Exercise 3.1

1. (a) $0.05e^{0.05x}$ (b) $1.2e^{1.2x}$
   (c) $-0.05e^{-0.05x}$ (d) $-1.1e^{-1.1x}$
   (e) $-0.02e^{-0.02x}$ (f) $-10e^{-10x}$
   (g) $0.5e^{0.01x}$ (h) $-0.09e^{-0.03x}$
   (i) $0.1125e^{-0.09x}$
2. (a) $2x + 3e^x$ (b) $-1/x^2 + e^{x/4}$
   (c) $2e^x(e^x + 1)$ (d) $4e^{4x} - 4e^{-4x}$
   (e) $4(x+1)^3 + e^{5x}$ (f) $-6x(1-x^2)^2 - 12e^{-3x}$
   (g) $-e^{-4x} - 2/(x+1)^3$ (h) $e^{2x} + 2/(1-x)^3$
   (i) $-e^{-x} - 2e^{-2x}$
3. (a) $2te^{-t} - t^2e^{-t}$ or $te^{-t}(2-t)$
   (b) $4te^{t/2} + t^2e^{t/2}$ or $te^{t/2}(4+t)$
   (c) $2te^t + (1+t^2)e^t$ or $(1+t)^2e^t$
   (d) $-3t^2e^t + (1-t^3)e^t$ or $e^t(1-3t^2-t^3)$
   (e) $-2(1-t)e^{-t} - (1-t)^2e^{-t}$ or $(1-t)(t-3)e^{-t}$
   (f) $4t(1+t^2)e^{2t} + 2(1+t^2)^2e^{2t}$
   or $2(1+t^2)(1+t)^2e^{2t}$
   (g) $2te^{-t} - t^2e^{-t}$ or $t(2-t)e^{-t}$
   (h) $-2t^{-3}e^{2t} + 2t^{-2}e^{2t}$ or $2t^{-3}e^{2t}(t-1)$
   (i) $[-e^{-t}(1+t)^2 - 2e^{-t}(1+t)]/(1+t)^4$
   or $-e^{-t}(t+3)/(1+t)^3$
4. (a) $f'(x) = 6x^2e^x + 2x^3e^x$
   (i) $40e^2$ (ii) 0, −3
   (b) $f'(x) = 3x^2e^{-x/2} - x^3e^{-x/2}/2$
   (i) $8/e$ (ii) 0, 6
5. (a) $f'(x) = 2xe^x + x^2e^x$,
   $f''(x) = 2e^x + 4xe^x + x^2e^x$
   $f'(x) = 0$ when $x = -2, 0$
   $f''(-2) = -2/e^2$, $f''(0) = 2$
   (b) $f'(x) = (2+x)e^x$, $f''(x) = (3+x)e^x$
   $f'(x) = 0$ when $x = -2$, $f''(-2) = 1/e^2$

Exercise 3.2

1. (a) $-2xe^{-x^2}$ (b) $(-2x+2)\,e^{-x^2+2x}$
   (c) $4(4x-1)e^{2x^2-x}$ (d) $x^2e^{x^3}$
   (e) $x^{-1/2}e^{x^{1/2}}$ (f) $-x^{-2}e^{x^{-1}}$
   (g) $-(1+2x)e^{-(x^2+x)}$
   (h) $-2x^{-1/2}e^{-x^{1/2}}$

2. (a) $2te^{t^2} + 2t^3e^{t^2}$ or $2te^{t^2}(1 + t^2)$
   (b) $e^{t^2+1} + 2t(1 + t)\,e^{t^2+1}$ or $e^{t^2+1}(1 + 2t + 2t^2)$
   (c) $(½)e^{-t^2} - (1 + t)te^{-t^2}$ or $(½)e^{-t^2}(1 - 2t - 2t^2)$
   (d) $[2t(1 + t)\,e^{t^2} - e^{t^2}]/(1 + t)^2$ or $e^{t^2}(2t^2 + 2t - 1)/(1 + t)^2$
3. (a) $P'(t) = 2te^{-t^2}$, $P''(t) = 2e^{-t^2} - 4t^2e^{-t^2}$
   $P''(t) = 0$ when $t = \pm 0.7071$; 0.8578, −0.8578
   (b) $Q'(t) = e^{-t^2} - 2t^2e^{-t^2}$
   $Q''(t) = -6te^{-t^2} + 4t^3e^{-t^2}$
   $Q''(t) = 0$ when $t = 0, \pm 1.2247$; 1, −0.4463, −0.4463

Exercise 4.1

1. (a) 0 (b) 1/2 (c) $3x^2$ (d) $6x - 2$ (e) $2e^{2x}$
2. (a) $d/dx(3x^4) = 12x^3$ (b) $d/dy(2y^3/3) = 2y^2$
   (c) $d/dx(4/(3x^2)) = -8/(3x^3)$
   (d) $d/dt(\sqrt{(2t)}) = 1/\sqrt{(2t)}$
   (e) $d/dx(e^{2x}) = 2e^{2x}$
   (f) $d/dx(1/e^x) = -1/e^x$
3. (a) 4 (b) 12 (c) −2 (d) 1/10 (e) e^{-3} (f) $3e^2$

Exercise 4.2

1. (a) $\sin(x), \cos(x)$ (b) $\cos(y), -\sin(y)$
   (c) $\tan(x), \sec^2(x)$
   (d) $x\sin(x), \sin(x) + x\cos(x)$
   (e) $t\cos(t), \cos(t) - t\sin(t)$
   (f) $\sin(\theta + \pi), \cos(\theta + \pi)$
2. (a) $\sqrt{2}/2$ (b) 0
   (c) $-\sin(2)$ (d) 4
   (e) 1 (f) 0
3. (a) $2\cos(2x)$ (b) $-¼\sin(x/4)$
   (c) $\cos(\pi/2 + x)$ (d) $-3\sin(3x)$
   (e) $½\sin(-x/2)$ (f) $\sin(\pi/4 - x)$
   (g) $5\sec^2(5x)$ (h) $[\sec^2(x/5)]/5$
   (i) $2\sec^2(\pi/4 + 2x)$
4. (a) $\cos(\theta + 1)$ (b) $-2\sin(2\theta + 1)$
   (c) $\sec^2(1 + \theta)$ (d) $\sin(1 - \theta)$
   (e) $¼\sec^2(\theta/4 + 1)$ (f) $¼\cos[(\theta + 2)/4]$
   (g) $-2\sin(\theta + 4)$ (h) $-1/2\cos(1 - \theta)$
   (i) 0
5. (a) $\sin[2(x + 1)]$
   (b) $-6\sin^2(1 - 2x)\cos(1 - 2x)$
   (c) $-9\cos^2(3x)\sin(3x)$
   (d) $-4x\sin[2(2x^2 + 1)]$
   (e) $-6\tan(1 - 3x)\sec^2(1 - 3x)$
   (f) $6x\tan^2(x^2 + 1)\sec^2(x^2 + 1)$
6. (a) $\sin(t) + t\cos(t)$
   (b) $2t[\cos(2t) - t\sin(2t)]$
   (c) $\tan(t^2) + 2t(1 + t)\sec^2(t^2)$
   (d) $\sin(t)[\sin(t) + 2t\cos(t)]$
   (e) $\sin(2t)[\sin(2t) + 4t\cos(2t)]$
   (f) $2t\cos^2(1 + 3t) - 3t^2\sin[2(1 + 3t)]$
7. (a) $\pi\cos(2\pi t)$ (b) $2\pi\tan(\pi t)\sec^2(\pi t)$
   (c) $2e^{2t}\cos(\pi t) - \pi e^{2t}\sin(\pi t)$
   (d) $-\pi e^{-\pi t}\cos(2t) - 2e^{-\pi t}\sin(2t)$
   (e) $-\pi e^{-\pi t}\sin(2t - \pi/6) + 2e^{-\pi t}\cos(2t - \pi/6)$
   (f) $2\pi e^{2\pi t}\cos^2(2t) - 4e^{2\pi t}\cos(2t)\sin(2t)$
8. (a) $[2t\cos(2t - 1) - \sin(2t - 1)]/t^2$
   (b) $[2t^2\sin(3 - 2t) - 2t\cos(3 - 2t)]/t^4$
   (c) $[-t\sin(t)e^{\cos(t)} - e^{\cos(t)}]/t^2$
   (d) $[\sin(t)\cos(t)e^{\sin(t)} - \cos(t)e^{\sin(t)}]/\sin^2(t)$
   (e) $2te^{\cos(\pi t)} - \pi t^2\sin(\pi t)e^{\cos(\pi t)}$
   (f) $[\sec^2(e^{\tan(t)})]\sec^2(t)e^{\tan(t)}$
9. (a) $3\sin^2(1 + t)\cos(1 + t)$;
   $3\sin(1 + t)[2\cos^2(1 + t) - \sin^2(1 + t)]$
   (b) $-8\cos^3(2t)\sin(2t)$;
   $16\cos^2(2t)[3\sin^2(2t) - \cos^2(2t)]$
   (c) $\sec^2(1 + t)$; $2\sec^2(1 + t)\tan(1 + t)$
   (d) $2\cos(2t)e^{\sin(2t)}$;
   $4e^{\sin(2t)}[\cos^2(2t) - \sin(2t)]$
   (e) $-e^t\sin(1 + e^t)$;
   $-e^t\sin(1 + e^t) - e^{2t}\cos(1 + e^t)$
   (f) $2\tan(t) + 2\tan^3(t)$; $2\sec^2(t)(1 + 3\tan^2 t)$

Exercise 5.1

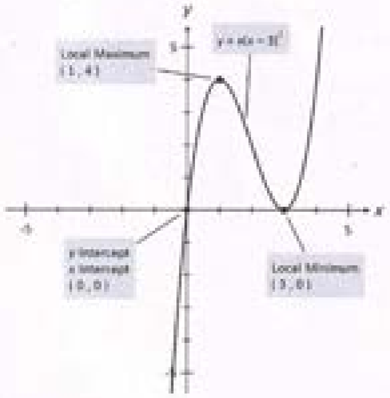
1. −10; $y = -10x + 9$
2. (a) $y = 4x/5 + 9/5$ (b) $y = -x/2 + 5/2$
   (c) $y = -5x/18 + 14/9$ (d) $y = 2e^2x - e^2$
3. $y = -2e^2x - e^2$; (−1/2, 0)
4. (a) $y = -x/(3e) + e + 1/(3e)$
   (b) $y = -x$
   (c) $y = -x/(2e) + e + 1/(2e)$
   (d) $y = (\pi/2)(x - \pi/2)$
5. (−1, 0)
6. (a) (0, 1) (b) (0, 0) & (2, −4)
   (c) (0, e) (d) ($\pi/2$, e)
7. (a) (0, 1) (b) (3, 2)
   (c) (−1, 1) & (−3, −9) (d) (0, 1)
8. $a = 1, b = 1$
9. $a = 1, b = 1$
10. $y = 2x$; (−3.1413, −6.2826) & (3.1413, 6.2826)

Exercise 5.2

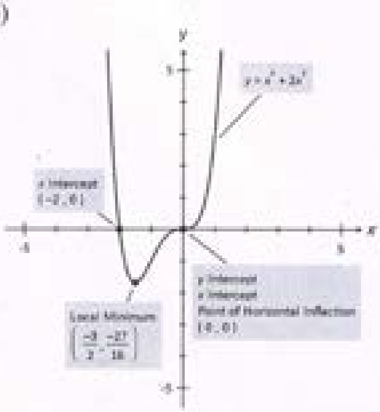
1. (a) (i) (1, 2) horizontal inflection point
   (ii) (1, 2) horizontal inflection point.
   (b) (i) (2, −2) horizontal inflection point
   (ii) (2, −2) horizontal inflection point.
   (c) (i) (1/3, 8/27) local maximum point
   (1, 0) local minimum point
   (ii) (2/3, 4/27) oblique inflection point.
   (d) (i) (0, 0) horizontal inflection point
   (−3/4, −27/256) local minimum point
   (ii) (−1/2, −1/16) oblique inflection point
   (0, 0) horizontal inflection point.
2. (a) $(-1, -1/e)$ local minimum point;
   $(-2, -2/e^2)$ oblique inflection point
   (b) $(1, e)$ local minimum point;
   No inflection points.
   (c) (0, 0) local maximum point;
   No inflection points.
   (d) No stationary points;
   (0, 0) oblique inflection points.
3. (2.0288, 1.8197) local maximum point;
   (0, 0) & (4.9132, −4.8145) local min. points.
4. $(-3\pi/4, -\frac{\sqrt{2}}{2}e^{-3\pi/4})$ local minimum point;

   $(\pi/4, \frac{\sqrt{2}}{2}e^{\pi/4})$ local maximum point;

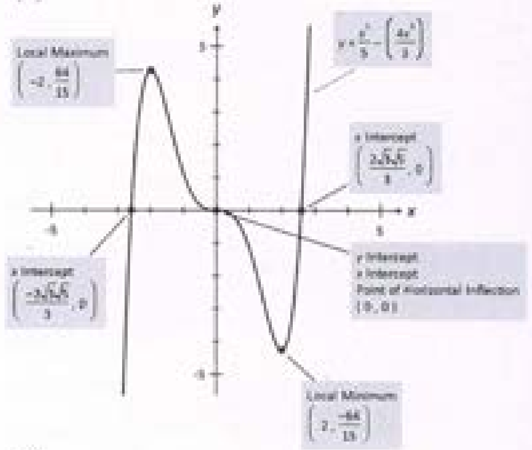
   $(0, 1)$ & $(\pi, -e^{\pi})$ oblique inflection points.
5. $a = 2/3$, $b = -6$
6. $a = -2/15$, $b = 4/5$
7. $a = 13/3$, $b = -25/3$
8. $b = 1$, a is any real number
9. (a)



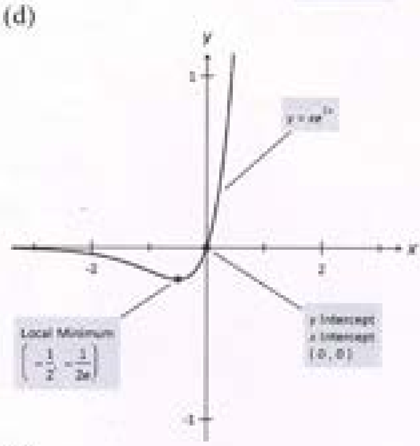
(b)



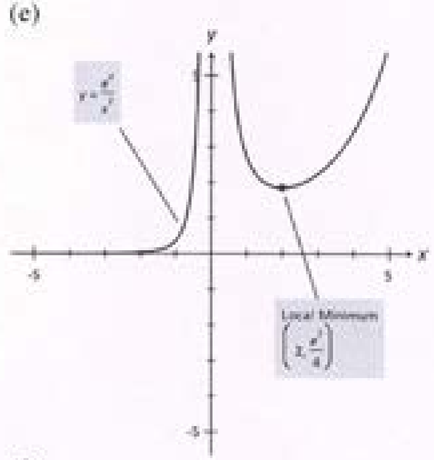
9. (c)



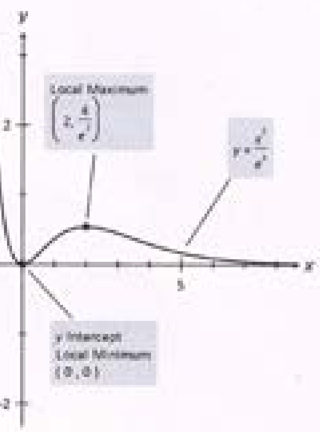
(d)



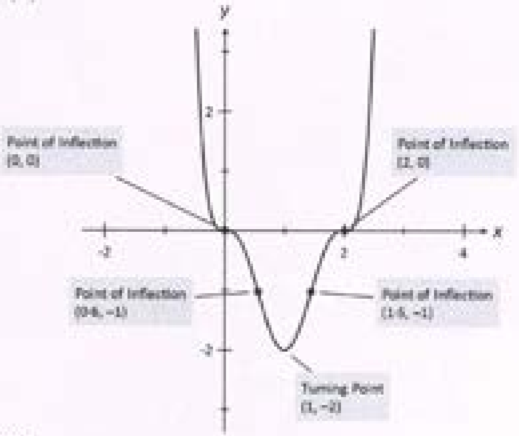
(e)



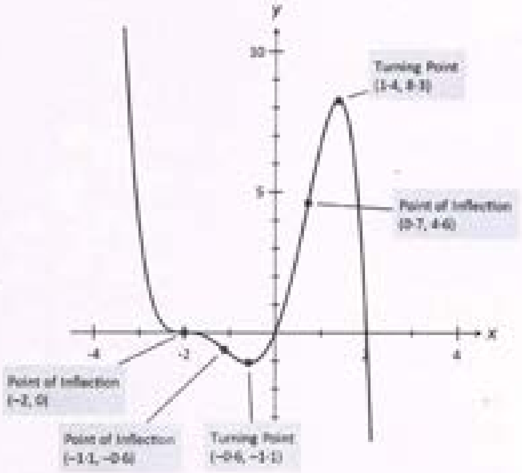
(f)



10. (a)



(b)



Exercise 6.1

1. (a) $3t^2 - 60t$
   (b) Max $y = 6000$, Min $y = 2000$
2. (a) $3000(100 - t^2)/(t^2 + 100)^2$
   (b) Max $P = 5150$
3. (a) Rate at which the no. infected changes with time
   (b) max $p = 0.25$ when $t = 2$
4. (a) $0.2e^{-t}[7 - 8t]$, 7/8
   (b) $1.6e^{-7/8}$, $t = 7/8$
5. (a) $10\,e^{-t}[\cos(t) - \sin(t)]$
   (b) $5(\sqrt{2})e^{-\pi/4}$, $\pi/4$
6. (a) $e^{-0.05x^2}[0.1x^2 - 1]$
   (b) $(\sqrt{10})e^{-0.5}$, $x = \sqrt{10}$
7. (a) 505, $t = 4\pi/3$ (b) 546 when $t = 6$.
8. 862 when $t = 0.8$; 339 when $t = 6.1$
9. (a) $(5\pi\sqrt{2})/12$ (b) 35 °C at 3 pm
   (c) 1 pm & 5 pm
10. (a) −0.2710 m/hour (b) 13 m at 6.30 am
    (c) 2.55 pm & 4.05 pm
11. (a) $5\pi/4$ cm/sec
    (b) 0.33, 3.67, 4.33 & 7.67 sec.
12. $(1/b)\ ln(ab/\lambda)$; $S < 0$ which is impossible
13. $\pi/6 = 30°$; $(3a\sqrt{3})/4$

Exercise 6.2

1. (a) $h = 100 - 5x$ (b) $V = 400x^2 - 20x^3$
   (c) 40/3 cm by 160/3 cm by 100/3 cm; 640 000/27 cm^3
2. (a) $L = 13x$ (b) $w = 500/3 - 20x$
   (c) $V = 5\,000x^2 - 600x^3$
   (d) 250/9 cm by 200/3 cm by 500/9 cm; 37 500 000/729 cm^3
3. (a) $L = 50 - 2x$
   (b) $V = 2x^3 - 90x^2 + 1\,000x$
   (c) 7.36 cm by 12.64 cm by 35.28 cm; 3 282.1 cm^3
4. Max volume of 8 455.8 cm^3; 7.85 cm by 24.31 cm by 44.31 cm
5. Max voume of $16\,000/\sqrt{\pi}$; radius $20/\sqrt{\pi}$, height $40/\sqrt{\pi}$
6. 17.8 cm by 35.6 cm by 23.7 cm
7. Min S = 4 606 cm^2; radius 15.6 cm, height 31.3 cm
8. Max area $562\,500/\pi$ m^2; circular playing field of radius $750/\pi$ m
9. $\theta = 2$ radians, max area 625 m^2
10. $20\sqrt{2}$ cm
11. $a = r\sqrt{2}$, $b = r\sqrt{2}$
12. $a = (r\sqrt{2})/2$, $b = (r\sqrt{2})/2$
13. 1/4 cm each
14. (c) $16\sqrt{x} + 64/x$ m^2
15. $h : r = 2:1$
16. 69.3 kmh^{-1}
17. $x = 225$ m, 8 min
18. Cindy should ride to a point on the dirt track such that $x = 2.18$km and then ride along the dirt track to T; 1 hr 27 min
19. 1.0921 radians
20. $[a^2/(3k^2)]^{1/4}$; $\pi/6$
21. (b) $4\pi r^2 \sin(x)\cos^2(x)$

Exercise 6.3

1. (a) 0.02 (b) −0.1
2. (a) −0.001 307 (b) 0.006 536
3. (a) 1.18 (b) −0.24
4. (a) −0.022 (b) 0.067
5. $0.32\pi = 1.01$ cm^2
6. $-10/[8\sqrt{(375\pi)}] = -0.036$ cm
7. $-\pi = -3.14$ cm^3
8. 0.03 cm
9. 0.71 cm^2
10. (a) $0.4\pi = 1.26$ cm^3 (b) $2\pi = 6.28$ cm^3
11. (a) $0.4\pi = 1.26$ (b) $1.4\pi = 4.40$

12. 0.05 m
13. (a) 3% (b) −6%
14. 3%, 2%
15. −2.5%, 4%

Exercise 6.4

1. \$6.20, \$6.20
2. \$32, \$32
3. \$20.04
4. \$3.63, \$12.26
5. $10x - 0.6x^2$, 8
6. \$1.40; $0.2x - 0.003x^2 - 1.4$; 59; 33
7. 133, 67
8. $x = 29$, \$0
9. \$12.03, $x = 3$
10. 833

Exercise 6.5

1. (a) 0.78% per year
   (b) $dP/dt = 0.1396e^{0.0078t}$
   (c) 150 946 persons per year
   (d) 46.1 years
2. (a) 116 goats (b) 8.2 goats per year
   (c) 13.9 years
3. (a) 780 koalas (b) −3.9 koalas per year
   (c) 138.63 years
4. (a) 13.53 g (b) 0.27 g per year
   (c) 0.005 4 g/year per year
5. (a) $P = 250e^{0.03t}$ (b) after 9.59 years
   (c) 9.59 years after 2006
6. (a) 179 (b) −32 persons per yr
   (c) 8.66 years after 2010
7. (a) 329 (b) 16.5 bacteria/hour
   (c) after 13.86 hours
8. (a) 0.19 g (b) 0.048 g per year
   (c) 0.012 g/year per year
9. (a) X: $dA/dt = -2.5e^{-0.05t}$
   Y: $dA/dt = -4.2e^{-0.07t}$
   (b) 25.94 years after 2010
10. (a) 27.85 years after 1981
    (b) 17.10 years after 1981

Exercise 7.1

1. (a) $-1/t + C$ (b) $-1/(6t^2) + C$
   (c) $-1/(54t^2) + C$ (d) $4/(3t^3) + C$
   (e) $-5t/(\pi^2) + C$ (f) $-4/t + C$
   (g) $-9/(4t) + C$ (h) $4/(27t^2) + C$
2. (a) $(8x^{3/2})/3 + C$ (b) $(2\sqrt{2}/3)x^{3/2} + C$
   (c) $-(15x^{4/3})/3 + C$ (d) $(3x^{4/3})/2 + C$
   (e) $-8x^{1/2} + C$ (f) $(x^{2/3})/2 + C$
   (g) $-(15x^{4/5})/2 + C$ (h) $(9x^{1/3})/2 + C$
3. (a) $-1/t + 1/(2t^2) + C$ (b) $-1/t^3 - 2t^{1/2} + C$
   (c) $-1/(2t)+2t^{3/2}/3+C$ (d) $-1/t - 1/(2t^2) + C$
   (e) $2t^{1/2} + 2t^{3/2}/3 + C$ (f) $-2/\sqrt{t} + 1/t + C$
   (g) $t^2 - 1/(2t^2) - 2/t + C$
   (h) $-16/(3t^3) - 4/t^2 - 1/t + C$
4. $f(x) = x + (4x^{3/2})/3 + x^2/2 - 68/3$
5. $y = x + 2/x - 1/(3x^3) - 2/3$
6. $v = 4t - 4/t - 1/(3t^3) + 13/3$

Exercise 7.2

1. (a) $(2 + x)^5/5 + C$ (b) $(3 + x)^6/6 + C$
   (c) $(1 + 2x)^4/8 + C$ (d) $-(3 + 4x)^4/16 + C$
   (e) $-(2 + 4x)^{3/2}/6 + C$
   (f) $-(1 - 3x)^{4/3}/4 + C$
   (g) $3(1 + 2x)^{5/3}/10 + C$
   (h) $-2(2 - 5x)^{5/2}/25 + C$
   (i) $3(2 + x)^{2/3}/2 + C$
   (j) $3(1 - 4x)^{2/3}/8 + C$
   (k) $2(1 + x)^{-1/2} + C$
   (l) $(2 + 3x)^{1/3} + C$
2. (a) $(x + 3)^4 + C$ (b) $(2x + 5)^3/24 + C$
   (c) $2/(2x + 3) + C$ (d) $1/[30(1 - 5x)^3] + C$
   (e) $8(1 - 3x)^{3/2}/9 + C$ (f) $-2(2 - 5x)^{3/2}/45 + C$
   (g) $(1 - 4x)^{1/2} + C$ (h) $(5 + 2x)^{1/2}/4 + C$
3. (a) $2x^2 + 2x^5/5 - 4x^3/3 + 2x + C$
   (b) $x - 4/x - 4/(3x^3) - (2x - 1)^4/8 + C$
4. $y = 4(2 + 3x)^5/15 - 113/15$
5. $P = 1/(1 + t) - 2$
6. $f(t) = 4(1 + 3t)^{1/2}/9 + 14/9$
7. $y = (x/2 + 4)^4/2 - 128$
8. $V = -8(0.5t + 4)^{1/2} + 16$

Exercise 7.3

1. (a) $\dfrac{(5+x^2)^4}{4} + C$ (b) $\dfrac{-(1-x^2)^8}{16} + C$
   (c) $\dfrac{(x^2+4)^6}{4} + C$ (d) $\dfrac{(3+2x^3)^5}{6} + C$
2. (a) $\dfrac{-5}{12(1+2x^2)^3} + C$ (b) $\dfrac{9}{16(3-4x^2)^2} + C$
   (c) $\dfrac{-1}{16(x^2-1)^2} + C$ (d) $\dfrac{-5}{84(2x^3+1)^2} + C$
3. (a) $\dfrac{-4}{3}(1+x^2)^{\frac{3}{2}} + C$
   (b) $\dfrac{1}{4}(1-x^2)^{\frac{3}{2}} + C$
   (c) $\dfrac{3}{2}(1+2x^2)^{\frac{1}{2}} + C$
   (d) $\dfrac{1}{6}(4x^2-5)^{\frac{1}{2}} + C$

4. (a) $\frac{-4}{7}(1-x^7)^{\frac{5}{4}}+C$

   (b) $\frac{15}{32}(2x^4-7)^{\frac{4}{5}}+C$

   (c) $\frac{1}{12}(1+2x+x^2)^6+C$

   (d) $\frac{1}{16(3-2x^2+x^4)^4}+C$

   (e) $\frac{-2}{3}(1+\frac{1}{x})^{\frac{3}{2}}+C$

   (f) $\frac{1}{4}(1+\sqrt{x})^8+C$

Exercise 7.4

1. (a) $e^{2x}/2+C$ (b) $-e^{4x}/4+C$
   (c) $50e^{0.06x}/3+C$ (d) $-20e^{0.05x}+C$
   (e) $500e^{0.02x}+C$ (f) $e^{4x}/40+C$
   (g) $10e^{0.4x}/3+C$ (h) $16e^{-1.25x}/25+C$
2. (a) $-e^{-3t}/3+C$ (b) $400e^{-0.01t}+C$
   (c) $10e^{-1.1t}/33+C$ (d) $-e^{-0.75t}+C$
   (e) $e^{2t}/2+C$ (f) $3e^{t}+C$
   (g) $-e^{4t}/8+C$ (h) $16e^{0.05t}+C$
3. (a) $e^{x+1}+C$ (b) $e^{3x-4}/3+C$
   (c) $e^{2x+1}+C$ (d) $e^{1+x}/2+C$
   (e) $-e^{-2x-1}/2+C$ (f) $e^{-1+2x}+C$
   (g) $-30e^{-0.05x+1}+C$ (h) $16e^{-1-0.05x}+C$
4. (a) $x^2+3e^{2x}/2+C$
   (b) $x^3-e^{-4x}/4+C$
   (c) $-1/x-e^{-3x+1}/3+C$
   (d) $-e^{0.5x+1}/2-1/x+C$
   (e) $(3x+1)^5/15-e^{-2x+2}/2+C$
   (f) $2e^{\frac{x}{2}}-\frac{1}{(x+1)}+C$
   (g) $\frac{-e^{-3x+1}}{3}+(1+2x)^{\frac{1}{2}}+C$
   (h) $\frac{(e^x+1)^3}{3}+C$
   (i) $\frac{-(1-e^{-2x})^5}{10}+C$
5. $V=-e^{-2t}/2+e^{2t}/2+2t$
6. $P=t-4e^{-t}-2e^{-2t}+7$

Exercise 7.5

1. (a) $e^{x^2}+C$ (b) $e^{-x^3}+C$
   (c) $e^{1-x^2}+C$ (d) $e^{x^2+x}+C$
   (e) $-e^{-x^2}+C$ (f) $-e^{-x^4}+C$
   (g) $-e^{1-2x^2}+C$ (h) $\frac{1}{3}e^{x^3+2x^2}+C$
2. (a) $\frac{1}{16}e^{1+2x^2}+C$ (b) $-\frac{1}{6}e^{x^3+2}+C$
   (c) $\frac{3}{4}e^{x^2}+C$ (d) $-\frac{4}{15}e^{x^3}+C$
   (e) $-\frac{1}{3}e^{-2x-x^2}+C$ (f) $-\frac{2}{3}e^{-x^2-2}+C$
3. (a) $\frac{(1+3e^{x^2})^4}{24}+C$ (b) $\frac{(1-e^{2x^2})^{-3}}{12}+C$
   (c) $\frac{11(1-e^{3x^2})^{\frac{-3}{2}}}{9}+C$
4. $3^{2x}/(2\ln 3)+C$
5. $2^{3x+1}/(3\ln 2)+C$

Exercise 7.6

1. (a) $-(\cos 2x)/2+C$ (b) $-2\cos t+C$
   (c) $(1/2)\cos(1-2x)+C$
   (d) $(-1/\pi)\cos(\pi x)+C$
   (e) $(3/4)\cos(4t/3)+C$
   (f) $(-1/12)\cos(4x)+C$
   (g) $(-3/4)\cos(4t-\pi/4)+C$
   (h) $(3/2)\cos((1-2x)/3)+C$
   (i) $3\cos((2x+\pi)/6)+C$
2. (a) $(1/3)\sin(3t)+C$ (b) $-4\sin(x)+C$
   (c) $(-1/\pi)\sin(1-\pi x)+C$
   (d) $(-3/4)\sin(-4t/3)+C$
   (e) $(1/10)\sin(5x)+C$
   (f) $2\sin(1+x/2)+C$
   (g) $(1/2)\sin(4t+\pi/3)+C$
   (h) $(-4/3)\sin((1+3x)/4)+C$
   (i) $(\sqrt{2})x/2+C$
3. (a) $(1/2)\sin(2x)-4\cos(x)+C$
   (b) $e^{4x}/4+3\cos(1-x)+C$
   (c) $-e^{-2x}+4\sin(x/4)+C$
   (d) $x^3-(\cos 2x)/2+C$
   (e) $(-1/4)\cos^4(x)+C$ (f) $(1/5)\sin^5(x)+C$
   (g) $(1/12)\sin^6(2x)+C$
   (h) $(1/4)\sin(4x)+C$ (i) $-\cos(x)+C$
   (j) $(1/3)\sin(3x)+C$
4. $\sin(x)-x\cos(x)+C$ 5. $(-1/2)\cos^2(x)+C$

Exercise 8.1

1. (a) 1/3 (b) 2
   (c) 16/3 (d) 112/3
   (e) 1/2 (f) 6/25
   (g) 19/2 (h) e

2. (a) $4/9$ (b) $14/3$
   (c) $3/2 - 1/e$ (d) $e^2 - e^{-1}$
   (e) $2/3$ (f) $-2(1 - \sqrt{2})$
   (g) $(e^4 - e)/2$ (h) $-3/4$
3. (a) $13/3$ (b) $-10/3$
   (c) $7/8$ (d) $1/24$
   (e) e (f) 2.6966
   (g) 10.4305 (h) $\frac{-5}{2}\left[\sqrt{3}-1\right]$
   (i) $-\left[\sqrt{1+e}-\sqrt{2}\right]$
4. (a) $3/4$ (b) 4 (c) $-3\pi^2/8+1$
   (d) 0 (e) 0 (f) $9/(2\pi) + e^4 - 1$
5. 1 6. $e^{\pi/2} + e^{-\pi/2}$
7. -1 8. $4/3$
9. $(1/2)e^{-\pi} + (1/2)$

Exercise 8.2

1. (a) $4x + x^2$ (b) $\sqrt{(1 + x^2)}$
   (c) $4xe^{x^2}$ (d) $-1/\sqrt{x}$
   (e) $8x^2(2x + 1)$ (f) $2t(t^2 - 1)/(t^2 + 1)$
   (g) $\frac{e^x}{8x}$ (h) $\sqrt{(1 + e^{x+5})} - \sqrt{(1 + e^x)}$
   (i) $x\cos(x)$ (j) $\sin(x)\sin(\sin x)\cos(x)$
   (k) $\frac{e^{\sin^2(u)}\cos(u)}{4\sin(u)}$ (l) $2\sqrt{1+e^{2x}}$
2. (a) $x^2 + 4x - 5$
   (b) $(\pi^2 + 4\pi) - (x^4 + 4x^2)$
   (c) x^2e^x (d) $e^x(e^{2x} + \sqrt{e^x})$
   (e) $2\sqrt{(1 + 4x^2)}$ (f) $2t(1 - t^2)/(1 + t^4)$
   (g) $\frac{e^{e^{2x}}}{e^{4x}} - e$ (h) $\sqrt{2} - \frac{1}{x}\sqrt{1+\frac{1}{x^2}}$
   (i) $\sqrt{[1 + (x + 2)^3]} - \sqrt{(1 + x^3)}$
   (j) $\sin^2(x)$ (k) $\tan^2(x)$
   (l) $e^{3x}\sin(e^x)$
3. $1/5$ 4. 1 5. 1
6. (a) $-2, 0$ (b) ± 1
   (c) ± 2 (d) 2

Exercise 8.3

1. (a) -10 (b) 9 (c) 21
   (d) -5 (e) -2.5 (f) 3
2. (a) 14 (b) -3 (c) $-33/2$
   (d) -19 (e) 2 (f) 1
3. (a) 1 (b) $1/3$ (c) 118
   (d) 5 (e) $-3/2$ (f) 10
4. (a) -23 (b) 0 (c) 0.6504
   (d) -2 (e) $\pi+(2/\pi)$ (f) -6.5332

5. (a) -30 (b) 40
   (c) 40 (d) 20
   (e) 10 (f) 20
6. (a) 74 (b) -10
   (c) $-10/3$ (d) 10
   (e) -20 (f) 20
7. (a) -28 (b) 12
   (c) 4 (d) -4
   (e) 4 (f) 8
8. (a) -37 (b) -10
   (c) -5 (d) $-5/2$
   (e) -10 (f) -5

Exercise 9.1

1. (a) $y = 2x^3 + x^2 - 5x + 2$
   (b) $y = e^x + x^2/2 + x + 1$
   (c) $y = -e^{-x} - x^2$
   (d) $y = 2\sqrt{x+2} - 2$
   (e) $y = \sqrt{x^2+1} - 1$
   (f) $y = 2x - \cos(x) + 2$
2. (a) $y = -1/x + 3$
   (b) $y = -1/(x + 1) + 2$
   (c) $y = 2(x + 1)^{3/2}/3 + 1/3$
   (d) $y = -1/[2(x^2 + 1)] + 3/2$
   (e) $y = 2\sqrt{x} - 1/e^x + 3$
   (f) $y = -\cos(\pi x)/\pi + \pi$
3. $y = (x^3/3 - x)$
4. $y = x^3/3 + x^2 - 1/3$
5. $y = x^2 + x + 1$
6. $y = -1.39e^x + 3.39x + 0.39$
7. $y = x^3/6 - x^2/2$

Exercise 9.2

1. (a) 2.35 (b) $14/3$ (c) $7/6$
   (d) $7/3$ (e) 3 (f) 2
   (g) 8 (h) $5/2$ (i) 4
2. (a) $9/2$ (b) $4/3$
   (c) $131/4$ (d) $71/6$
3. (a) 2 (b) 4
   (c) $37/6$ (d) $75/4$
4. (a) 4.53 (b) 3
   (c) 0.22 (d) $2\pi = 6.28$
5. (a) $8(\sqrt{2})/3$ (b) $8(\sqrt{2})/3 - 4/3$
   (c) $5/3$ (d) $4(\sqrt{2})/3 - 4/3$

Exercise 9.3

1. (a) 1 (b) 3
   (c) $5/6$ (d) $9/2$
   (e) $9/8$ (f) $37/24$
   (g) $47/24$ (h) $125/24$
   (i) $37/24$
2. (a) $4/27$ (b) $32/27$
   (c) $11/12$ (d) $37/12$

2. (e) 3/4 (f) 7/12
   (g) 3.12 (h) 7.54
   (i) 6.96
3. (a) 9/2 (b) 9/8
   (c) 125/24 (d) 37/12
   (e) 4/3 (f) 7.54
   (g) 8/3 (h) 1/6
   (i) 9/2 (j) 32/15 = 2.1333
4. (a) $1 + 3\pi/2$ (b) $\sqrt{2}$
5. (a) 60 m (b) 7000/3 = 2333.3 m^2
6. (a) 68/3 m^2 (b) 680 kL
7. 55.79 m^2 8. 2 m^2
9. 1.27 cm^2

Exercise 9.4

1. (a) $-t^2/2 + 2t$; 0, 2 (b) 0, 4
   (c) local max of 2
2. (a) $t^2 - 4t - 32$; –20, 0
   (b) –2, 6
   (c) Global min. –36; Global max. 0
3. (a) $10 - 2t$ (b) 6, 6
   (c) $F'(2) = f(2)$
4. (a) $t^2 - 4$ (b) 12, 12
   (c) $F'(4) = f(4)$
6. Global min. 4175/12; Global max. 9295/12
7. Global min. –2390.4; Global max. 0

Exercise 9.5

1. $A = 2t^2 + 2t + 1$ 2. 28/3
3. 5.2732 4. 0.4908
5. 2 6. 2.4721
7. 2/15; 1/15
8. (a) $t = 40$ min
   (b) (i) 9.375 °C (ii) 1.775 °C
9. (a) P_{min} = 14.17 cents when $t = 5$ months
   (b) (i) –42.00 cents (ii) 12.83 cents
10. (a) 1/3 mg/kL
    (b) C_{max} = 10.05 mg/kL when $t = 1/2$ week
11. (a) $t = 4$ hours (b) 18.25 kL (c) 88 kL
12. (a) –1.81 volts (b) 35.27 min
13. (a) $220 (b) $600
14. (a) $400 (b) $1 710
15. (a) Liquid starts flowing out of the tank
    (b) Net increase in first minute 2/3 L
    (c) Net decrease in second minute 4/3 L
    (d) 13/3 L
16. 1435.66, 28.71 17. –2, –180, –20
18. 26.87, 99173.55, 826.45
19. $-3bF_o/(4F_o + a)$

Exercise 10.1

1. (a) 4 m
   (b) $t = 1, v = -3$ ms^{-1}; $t = 4, v = 3$ ms^{-1}
   (c) 2.5 s (d) 3 seconds
2. (a) t = 1, 3, 4 seconds
   (b) t = 1.78, 3.55 seconds
   (c) 2 seconds

2. (d) 5.3 ms^{-2}, –5.3 ms^{-2}
   (e) 16.2 m
3. (a) –8 ms^{-1} (b) 2 ms^{-2}
   (c) 2 ms^{-2}; body experiences constant acceleration
4. (a) t = 2 or 6 seconds
   (b) –4 ms^{-1} (c) –4 ms^{-2}, 4 ms^{-2}
   (d) 32 ms^{-1} when t = 10 seconds
5. (a) t = 1, 4 seconds (b) 1 ms^{-1}
   (c) –3 ms^{-1} (d) 6 m
   (e) 3 ms^{-1}
6. (a) 4 ms^{-1}, –2 ms^{-2} (b) –0.73 ms^{-2}
   (c) –1 ms^{-2} (d) 1 ms^{-1}
   (e) velocity tends to 0 ms^{-1}
7. (a) $10/e$ = 3.68 ms^{-1} (b) –0.55 ms^{-2}
8. (a) $3\pi/4$ s, –14.92 ms^{-1} (b) 4.81 ms^{-1}
9. (a) 0.4637 + $(n\pi/2)$ seconds n = 0,1,2,3,.....
10. (a) 12.5 ms^{-1} (b) 0.93 ms^{-1}
    (c) 4.63 m (d) 0.93 ms^{-1}
    (e) P comes to rest.
11. (a) 19.6 ms^{-1} (b) 2 seconds
    (c) t = 4 seconds, –19.6 ms^{-1}
    (d) –9.8 ms^{-2}; acceleration is constant
12. (a) –9.8 ms^{-1} (b) –29.4 ms^{-1}
    (c) –9.8 ms^{-2}
    (d) –9.8 ms^{-2}; acceleration is constant
13. (a) 10 ms^{-1} (b) 5.10 m
    (c) –10.0 ms^{-1} (d) –9.8 ms^{-2}
14. (a) 27.5 m (b) –30.0 ms^{-1}
    (c) 1.02 s (d) 1.02 s, 5.10 s
15. (a) 1.33 s (b) 4.55 m
    (c) 2.32 ms^{-1}, –2.32 ms^{-1}
    (d) 8.01 ms^{-1}
16. (a) 5 ms^{-1} (b) 6.25 m
    (c) –2 ms^{-2}; constant acceleration
17. (a) 6 s
    (b) Stopping distance is 36 m, so Yes!
    (c) –2 ms^{-2}; constant acceleration
18. (a) 14 ms^{-1} (b) 2.45 s
    (c) $2t$ ms^{-2}; variable acceleration
    (d) 19.33 ms^{-1}

Exercise 10.2

1. (a) 10 ms^{-1} (b) -2 ms^{-2}
   (c) 50 m (d) 100 m (e) 20/3 ms^{-1}
2. (a) 0 ms^{-1} (b) $t = 5$, 15 seconds
   (c) -1 ms^{-2} (d) 50 m
   (e) P is at the starting point.
3. (a) 38/3 m (b) 6 ms^{-2}
4. (a) 11/6 m (b) 5 ms^{-2}
5. (a) 0 (b) 0 (c) 4 m (d) 2 ms^{-1}
6. (a) -2π ms^{-1} (b) 0 (c) 6 m (d) 4 ms^{-1}
   (e) $-8\pi/3$ ms^{-2}
7. (a) 1.98 ms^{-1} (b) 7.02 m
   (c) 7.02 m (d) 7.02 m
8. (a) -1.5 ms^{-1}
   (b) P moves with a constant velocity of -4 ms^{-1}
   (c) -1.77 m (d) 6.23 m (e) $-5/2$ ms^{-2}
9. (a) 2 ms^{-1} (b) 0.38 m
   (c) 0 m (d) 1.54 m (e) 0 ms^{-2}
10. (a) 0.45, 2.22 s (b) 2.11 m
    (c) 0 m (d) 5.49 m (e) 1.83 ms^{-1}
11. $k = 3$
12. $k = -2$, $C = 6$

Exercise 11.1

1. (a) No; $\Sigma p(x) \neq 1$
   (b) Yes; $p(x) \geq 0$ for all x and $\Sigma p(x) = 1$
   (c) No; $\Sigma\, p(x) \neq 1$
2. (a) No; $\Sigma\, f(x) \neq 1$
   (b) Yes; $f(x) \geq 0$ for all x and $\Sigma\, f(x) = 1$
3. (a) Yes; $p(x) \geq 0$ for all x and $\Sigma\, p(x) = 1$
   (b) Yes; $p(x) \geq 0$ for all x and $\Sigma\, p(x) = 1$
   (c) Yes; $p(x) \geq 0$ for all x and $\Sigma\, p(x) = 1$
   (d) Yes; $p(x) \geq 0$ for all x and $\Sigma\, p(x) = 1$
4. (a)

x	0	1	2	3	4
$P(X \leq x)$	0.1	0.2	0.4	0.6	1.0

   (b) 0.4 (c) 2/3 (d) 0.5
5. (a) $k = \frac{1}{10}$
   (b)

x	0	1	2	3	4
$P(X \leq x)$	0	0.1	0.3	0.6	1.0

   (c) 0.2 (d) $\frac{2}{3}$
6. (a) $k = \frac{12}{25}$
   (b)

x	1	2	3	4
$P(X \leq x)$	12/25	18/25	22/25	1

   (c) $\frac{7}{25}$ (d) $\frac{4}{7}$
7. (a) $k = 10$
   (b)

x	-2	-1	0	1	2
$P(X \leq x)$	0.4	0.5	0.5	0.6	1.0

   (c) 0.5 (d) $\frac{1}{6}$
8. (a) $0 \leq k \leq \frac{1}{10}$ (b) $\frac{1-4k}{1-3k}$
9. (a) $k = 20$ (b) $\frac{5}{18}$
10. (a) $5e^{-2}$ (b) 5/7

Exercise 11.2

1. (a) 9/2, $(\sqrt{21})/2$ (b) 7/3, $(\sqrt{5})/3$
   (c) 5/4, $(\sqrt{15})/4$ (d) 5/3, $(\sqrt{10})/3$
2. (a) 3.4, 1.4967 (b) 0.6 (c) 0.5
3. (a) 1, 1.6733 (b) 0.65 (c) 0.5
4. (a) 1/140 (b) 28/5, $(\sqrt{51})/5$
5. (a) 5, 5 (b) 4 or 5
6. (a) 24/13, $3(\sqrt{14})/13$ (b) 2
7. (a) $a = 0.3$, $b = 0.1$ (b) 0.7
8. (a) $a = 0$, $b = 5$ (b) $k = 3$
9. $a = 1/55$, $k = 10$
10. $a = 1/100$, $k = 4$
11. (a) $a = 0.35$ (b) 0 (c) \$8000
    (d) $b <$ \$2000
12. (a) $1500 + 0.4a + 0.3b$
    (b) $a = -\$15\,000$, $b = \$15\,000$
    (c) $a > -(0.75b + 3750)$
13. (a) \$130 750 (b) $k = -\$107\,500$
14. (a) 0.585 (b) Profit of 40 cents/game
    (c) Loss of \$1.40 cents/game
    (d) Charge of \$2.60/game
15. (a) 0.49 (b) Profit of \$1.58/game
    (c) Loss of 58 cents/game
    (d) Charge of \$1.50/game
16. (a)

x	0	1	2
$P(X = x)$	0.25	0.5	0.25
$P(X \leq x)$	0.25	0.75	1.0

    (b) Most like value = 1, Mean value = 1
17. (a)

x	0	1	2	3
$P(X = x)$	6/36	10/36	8/36	6/36
$P(X \leq x)$	6/36	16/36	24/36	30/36

x	4	5
$P(X = x)$	4/36	2/36
$P(X \leq x)$	34/36	1

    (b) Most like value = 1, Mean value = 35/18
18. (a)

x	0	1	2	3
$P(X = x)$	1/35	12/35	18/35	4/35
$P(X \leq x)$	1/35	13/35	31/35	1

    (b) Most like value = 2,
    Mean value = 12/7, std. dev. = $2(\sqrt{6})/7$
19.

x	0	1	2	3
$P(X = x)$	8/27	12/27	6/27	1/27

    X = 1 is the most likely since it has the highest probability

20. (a) $P(X = x) = \begin{cases} \frac{7}{48} & x = 1,2,3,4,5,6 \\ \frac{1}{16} & x = 7,8 \end{cases}$; $\frac{31}{34}$
    (b) Mean value = 4, std. dev. = (√39)/3
    Most likely value = 1, 2, 3, 4, 5, 6
21. (a) $P(X = 0) = \frac{\binom{8}{3}}{\binom{10}{3}} = \frac{7}{15}$

    $P(X = 1) = \frac{\binom{2}{1}\binom{8}{2}}{\binom{10}{3}} = \frac{7}{15}$

    $P(X = 2) = \frac{\binom{2}{2}\binom{8}{1}}{\binom{10}{3}} = \frac{1}{15}$;

    (b) $P(T = 3) = P(T = 7) = \frac{7}{15}$, $P(T = 11) = \frac{1}{15}$;
    Most likely no. of boxes is 3 or 7.
22. $P(X = 0) = \frac{\binom{6}{0}\binom{39}{6}}{\binom{45}{6}} = 0.400\ 564\ 637$

    $P(X = 1) = \frac{\binom{6}{1}\binom{39}{5}}{\binom{45}{6}} = 0.424\ 127\ 262$

    $P(X = 2) = \frac{\binom{6}{2}\binom{39}{4}}{\binom{45}{6}} = 0.151\ 474\ 022$

    $P(X = 3) = \frac{\binom{6}{3}\binom{39}{3}}{\binom{45}{6}} = 0.022\ 440\ 596$

    $P(X = 4) = \frac{\binom{6}{4}\binom{39}{2}}{\binom{45}{6}} = 0.001\ 364\ 631$

    $P(X = 5) = \frac{\binom{6}{5}\binom{39}{1}}{\binom{45}{6}} = 0.000\ 028\ 729$

    $P(X = 6) = \frac{\binom{6}{6}\binom{39}{0}}{\binom{45}{6}} = 0.000\ 000\ 123$

    Most likely no. of correct guesses is one as this has the highest probability.
23. $P(T = 13) = \frac{1}{6}$, $P(T = 14) = \frac{1}{2}$, $P(T = 15) = \frac{1}{3}$
    A&D = B & D, A & C = B & C
24. P(T = 0) = 0.1, P(T = 1) = 0.09,
    P(T = 2) = 0.081, P(T = 3) = 0.0729,
    P(T = 4) = 0.06561, P(T = 6) = 0.59049.
    Most likely value for T is 6.
25. (a) $P(N = k) = 0.01 \times 0.99^{k-1}$ for k = 1, 2, 3
    (b) 100
26. (a) $P(X = 1) = \frac{10}{36}$, $P(X = 2) = \frac{9}{36}$,
    $P(X = 3) = \frac{6}{36}$, $P(X = 4) = \frac{5}{36}$,
    $P(X = 5) = \frac{2}{36}$, $P(X = 6) = \frac{1}{36}$
    $P(X = 8) = \frac{1}{36}$, $P(X = 10) = \frac{1}{36}$,
    $P(X = 12) = \frac{1}{36}$
    (b) Most likely value = 1 (c) 28/9
27. (a) 3, 1 (b) E(F) = 3, Var(F) = 4

f	−1	1	3	5
P(F = f)	1/10	2/10	3/10	4/10

    (c) E(G) = $3a + b$, Var(G) = a^2

g	$a+b$	$2a+b$	$3a+b$	$4a+b$
P(G = g)	1/10	2/10	3/10	4/10

    (d) E(G) = aE(X) + b, Var(G) = a^2 Var(X)

Exercise 11.3

1. $p(x)$ = 1/5 for x = 0, 1, 2, 3, 4
   E(X) = 2, Var(X) = 2
2. E(X) = 1, Var(X) = 4
3. E(X) = n/2, Var(X) = $n(n + 2)$/12
4. E(X) = 7/2, STD(X) = (√105)/6
5. E(X) = 9/2, STD(X) = (√21)/2
6. (a) $p(x)$ = 1/6 for x = −2, −1 0, 1, 2, 3
   E(X) = 1/2, STD(X) = (√105)/6
   (b) (i) P(Particle has a positive charge) = 1/2.
   (ii) P(Particle has a clockwise spin) = 1/3
   (c) 1/2
7. (a) $p(x)$ = 1/1000 for x = 0, 1, 2, ….. , 999
   (b) E(X) = 999/2, STD(X) = 288.6750
   (c) 289/500
8. (a) $p(x)$ = 1/21 for x = 1, 2, 3, ….., 21
   (b) Var(X) = 110/3
9. (a) $p(x)$ = 1/11 for x = 1, 2, 3, ….., 10
   (b) Var(X) = 10
10. n = 4 000

Exercise 11.4

1. (a) $p(y)$ = 1/6 for y = 3, 5, 7, 9, 11, 13
   (b) E(Y) = 8, STD(Y) = (√105)/3
2. (a) $p(y) = (4 - y)$/15 for y = −1, 0, 1, 2, 3
   (b) E(Y) = 1/3, STD(Y) = (√14)/3
3. (a) 1/5, 2/9 (b) 3/40, (√7)/120
4. (a) 528/625, 33/34 (b) 18, 2√6
5. (a) 209/210, 185/209 (b) −4/5, 8/5
6. (a) 6, 4 (b) 6, 4
7. a = 3, b = 100; a = −3, b = −20
8. a = 1/2, b = 20; a = −1/2, b = 100
9. (a) 2.04, 0.32 (b) 1.694, 0.352
   (c) 1.309, 0.272 (d) 1.67, 0.16
10. (a) 36, 3.5 (b) 244/3, 36/5
    (c) 130/3, 35/12
11. (a) 55/16, −5 (b) 16, 30.08
12. 54 13. 8 14. 42, 5
15. (a) −5/23, −2/15 (b) 46.4, 47.8; 2014
16. More likely that P(Y ≥ 65) > P(X ≥ 72).

Exercise 12.1

1. (a) 0.0896 (b) 0.9502 (c) 0.9832
   (d) 0.9823 (e) 0.9493 (f) 3.2; 3
2. (a) 0.05481 (b) 0.4770 (c) 0.9916
   (d) 0.9824 (e) 0.4726 (f) 3.6; 4
3. (a) $4p(1-p)^3$ (b) $(1-p)^3(1+3p)$
   (c) $p^3(4-3p)$ (d) $4p$; $4p(1-p)$
4. (a) $126p^4(1-p)^5$ (b) $(1-p)^8(1+8p)$
   (c) p^9 (d) $9p$; $9p(1-p)$
5. (a) $\binom{n}{3}0.2^3\,0.8^{n-3}$ (b) 0.8^n
   (c) 0.2^n (d) $0.2n$; $0.16n$
6. (a) $\binom{10}{k}0.35^k\,0.65^{10-k}$
   (b) $\binom{10}{k+1}0.35^{k+1}\,0.65^{9-k}$
   (c) $\binom{10}{k}0.35^k\,0.65^{10-k}$
7. $n = 12$ 8. $n = 30$
9. (a) $n \le 6$ (b) $n \le 6$
10. (a) $n \le 91$ (b) $n \le 53$
11. (a) $n \ge 12$ (b) $n \ge 20$
12. (a) $n \ge 24$ (b) $n \ge 7$
13. $n = 100, p = 0.23$
14. (a) $n = 70, p = 0.4$ (b) $n = 140, p = 1/7$

Exercise 12.2

1. 0.12, 0.324 96 2. 0.27, 0.443 96
3. X_i is a Bernoulli variable with $p = 0.2$.
   Mean = 0.2, Variance = 0.16
   Y_i is a Binomial variable with $n = 50, p = 0.2$.
   Mean = 10, Variance = 8
4. X_i is a Bernoulli variable with $p = 0.95$.
   Mean = 0.95, Variance = 0.0475
   Y_i is a Binomial variable with $n = 100, p = 0.95$
   Mean = 95, Variance = 4.75
5. $P(X = x) = \binom{20}{x}0.65^x\,0.35^{20-x}$
   where $x = 0, 1, 2, 3, \ldots, 20$,
   Mean = 13, Variance = 4.55
6. X_i are sum of non-independent Bernoulli variables.
7. X_i are sum of non-independent Bernoulli variables.
8. (a) $X \sim B(20, 0.7)$
   (b) (i) 0.6080 (ii) 0.9874 (c) 14
9. (a) 0.0467 (b) 0.4369 (c) 0.1941
   (d) 0.9748
10. (a) 0.6646 (b) 0.9753
11. (a) 0.9496 (b) 0.8244
12. (a) 0.8188 (b) 0.8562 (c) 4.06; 1.6978
13. (a) 0.01977 (b) 0.8870
14. (a) 0.1901 (b) 0.01091 (c) 3
15. (a) 0.01198 (b) 0.07247
16. (a) 0.1298 (b) 0.9901
    (c) 0.009 866 (d) 0.002503
17. (a) 0.5695 (b) 0.2620 (c) 0.1784
18. (a) 0.01916 (b) 0.5348 (c) 0.0744
    (d) 14; 26.74
19. (a) 0.09515 (b) 4 (c) 16
20. (a) $P(X = x) = \binom{5}{x}0.2^x(0.8)^{5-x}$ for $x = 0,1,2,3,4,5$
    (b)

t	$P(T = t)$
25	0.00032
19	0.0064
13	0.0512
7	0.2048
1	0.4096
0	0.32768

    (c) 0.05792 (d) 11% (2.64 marks)
21. (a) $P(M = m) = \binom{6}{m}0.1^m\,0.9^{6-m}$ $m = 0,1,2,\ldots,6$
    (b)

r	$P(R = r)$
2400	0.531 441
1900	0.354 294
1400	0.098 4150
900	0.014 580
400	0.001 215
−100	0.000 054
−600	0.000 001

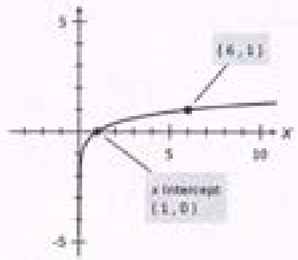
    (c) 0.000 055 (d) \$2100

Exercise 13.1

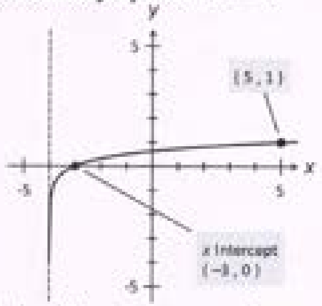
1. (a) $9 = 3^2$ (b) $1\,000 = 10^3$
   (c) $625 = 25^2$ (d) $2 = 4^{1/2}$
2. (a) $\log_{10}10\,000 = 4$ (b) $\log_2 64 = 6$
   (c) $\log_2(1/4) = -2$ (d) $\log_{81}9 = 1/2$
3. (a) 2 (b) 3 (c) 5 (d) 3
   (e) −1 (f) −2 (g) −1 (h) −4
4. (a) 3 (b) −80 (c) 24 (d) $\pm\sqrt{1001}$
   (e) no solution (f) $x = 1$

Exercise 13.2

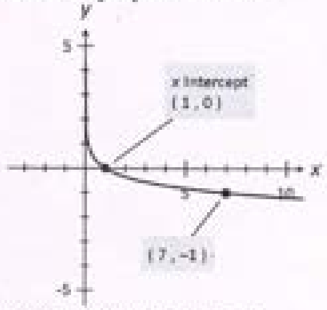
1. (a) $\dfrac{\log 56}{\log 4}$ (b) $\dfrac{\log 50}{\log 3}$
   (c) $\dfrac{\log(1/12)}{\log 5}$ (d) $\dfrac{\log(4/9)}{\log 2}$
   (e) $\dfrac{\log 3.4}{\log 7}$ (f) $\dfrac{\log(21/13)}{\log 11}$
2. (a) IV (b) I (c) V (d) II
3. (a) Vertical Asymptote: $x = 0$



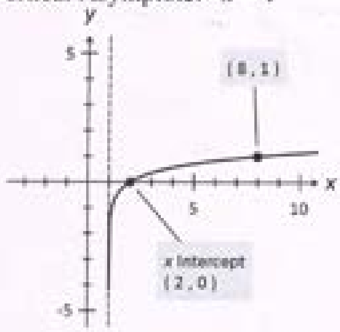
3. (b) Vertical Asymptote: $x = -4$



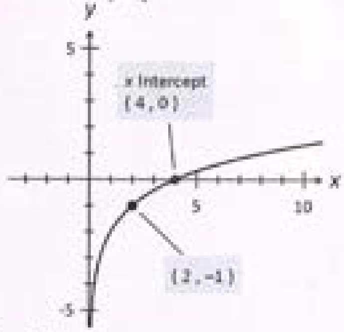
(c) Vertical Asymptote: $x = 0$



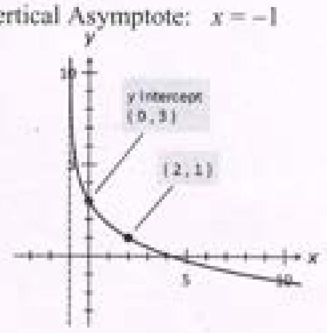
(d) Vertical Asymptote: $x = 1$



(e) Vertical Asymptote: $x = 0$



(f) Vertical Asymptote: $x = -1$



Exercise 13.3

1. (a) $1 + \log_5 x$ (b) $2 + \log_5 x$
   (c) $1 - \log_5 x$ (d) $-[1 + \log_5 x]$
   (e) $2\log_3 x + \log_3 y$ (f) $\log_3 x - 2\log_3 y$
   (g) $\log_2 x + \frac{1}{2}\log_2(1 + x)$
   (h) $\log_2 x - \frac{1}{2}\log_2(1 - x)$
2. (a) $5\log_5 x$ (b) x
   (c) $2 + \log_3 x$ (d) $2x$
   (e) $-2x$ (f) $2(1 + x)$
   (g) $-2(x + 1)$ (h) $x/2$
3. (a) $\log [x(1 + x)]$ (b) $\log (xy)$
   (c) $\log [y(1 + x)]$ (d) $\log (1 - x^2)$
   (e) $\log [x/(1 + x)]$ (f) $\log (x/y)$
   (g) $\log [(1 + x)/y]$ (h) $\log (1 - x)$
4. (a) $\log x^3$ or $3\log x$ (b) $\log (xy^2)$
   (c) $\log [x/(x + 1)^{1/2}]$ (d) $\log [y^{1/2}/(x + 1)]$
   (e) $\log [(x + 1)(1 - x)^{1/2}]$
   (f) $\log [(x + 1)^{1/2}/(1 + x^2)^2]$
   (g) $\log [x^{1/2}(1 + x)/(1 - x)^2]$
   (h) $\log [z^2/(x^{1/3}y^{1/2})]$
5. (a) 4/5 (b) −3/2 (c) $x/3$ (d) 2
6. (a) $\log_3(3x)$ (b) $\log_5(x/5)$
   (c) $\log_5(25x)$ (d) $\log_9(x/3)$
7. (a) $p + q$ (b) $p + 2q$
   (c) $p - q$ (d) $p + 1$
   (e) $p - 1$ (f) $3 + p + q$
8. (a) $\log_3 75$ (b) $\log_3 375$
   (c) $\frac{1}{2}\log_3 75$ (d) $\log_3 45$
   (e) 2 (f) 3
9. (a) 34 (b) ±9
   (c) $-1 + 5\sqrt{5}$ (d) 2
   (e) 3 (f) 2

Exercise 13.4

1. (a) (log 15)/(log 2) (b) 1/(log 3)
   (c) $x >$ (log 15)/(log 5)
   (d) $x \geq$ (log 5)/(log 2)
   (e) −1 + (log 6)/(log 2)
   (f) 1 + 1/(log 5)
   (g) −1 −(log 7)/(log 3)
   (h) −1 + 2/(log 4)
2. (a) (log 6)/(log 2) (b) 1/(log 3)
   (c) $x <$ (log 5)/(log 8) (d) $x \geq$ −(log 2)/(log 3)
   (e) −1 + (log 7)/(log 2)
   (f) [1 + (log 8)/(log 5)]/2
   (g) [1 + (log 3)/(log 7)]/2
   (h) 1 − (log 4)/(log 5)
3. (a) 1.2851 (b) 0.6712
   (c) 4.0875 (d) −1.2619
4. (a) (log 3)/(log 3 − log 2)
   (b) −(log 2)/(log 2 − log 5)
   (c) −6/5 (d) 2/(1 + log 5)
   (e) (log 3)/(log 4 − log 5)
   (f) (log 5 − log 4)/(log 3 − log 2)
   (g) (log 3 + 2log 7 − 2log 2)/(log 2 + log 7)

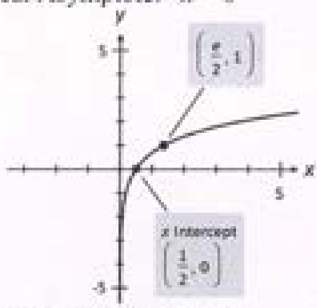
4. (h) (log 3 + log 7 − log 4 + log 5)/(log 5 + 2log 7)
   (i) 25 (j) −25 log (2)/(log 3)
5. (a) 2 (b) (log 5)/(log 2)
   (c) (log 7)/(log 3) (d) (log 5)/(log 3)
   (e) 0, 3 (f) 0, 1/(log 5)
   (g) 3 (h) −4
6. (a) (log 4)/(log 3), (log 2)/(log 3)
   (b) 1 (c) −1 (d) −1

Exercise 13.5

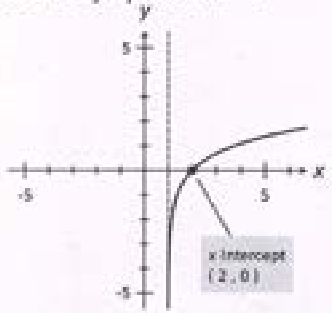
1. 0.00, 0.15, 0.30, 0.45, 0.60, 0.75, 0.90, 1.04, 1.20, 1.34, 1.51, 1.65
   The log (*f*-stop numbers) are approximately uniformly spaced out
2. (a) between 5.73 and 6.20
   (b) between 7.73 and 8.34
   (c) $10^{5.4} \approx 251\ 188$
3. (a) $10^{1.25} \approx 18$ times stronger
   (b) (i) 8.55 (ii) 7.95
4. (a) 7 (b) 2.0×10^{-9} moles/L
   (c) $10^{6} = 1\ 000\ 000$
5. (a) 202 (or 203) (b) 119 226 (or 119 227)
6. (a) 1.39 ms^{-1} (b) 20.04 s
   (c) 1.66 s and 145.28 s
7. (a) −1.85 m (b) 32.15 s
   (c) 0.13 m (d) 1.98 m
8. (a) $2x + x \log (x + 1)$ (b) $19.95x$
   (c) $P = 17.95x - x \log (x + 1)$
   (d) \$139 500 (e) 11 521
9. (a) 0.25 (b) 892
   (c) between 250 and 892
10. (a) 36 million for both courses
    (b) on the 14th day ($t = 13.54$)
    (c) on the 10th day ($t = 9.31$)
    (d) In treatment A, the bacteria population is constantly being reduced and is wiped out after 15 days. In treatment B, the bacteria population first increases, then decreases and is wiped out after 12 days. Treatment B is more effective in terms of no. of days required to wipe out the bacteria. But the increased levels in the earlier days could have fatal consequences.

Exercise 14.1

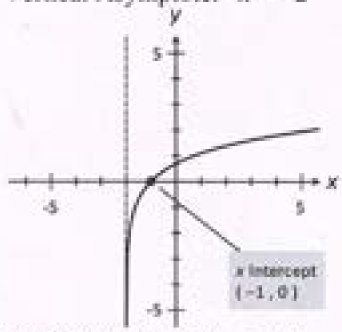
1. (a) 10 (b) x^2
   (c) $1/(1 - x)$ (d) $x^3(x - 2)$
   (e) x^2 (f) $(x - 1)$
   (g) $1 - x$ (h) x
   (i) 5 (j) 7
   (k) $(x + 5)$ (l) $1 - 2x$
2. (a) $e^{\ln 3}$ (b) $e^{-\ln 4}$
   (c) $e^{x \ln 5}$ (d) $e^{(\ln x)/2}$
3. (a) $2^{(\ln 3)/(\ln 2)}$ (b) $2^{(-\ln 5)/(\ln 2)}$
   (c) $2^{(2\ln x)/(\ln 2)}$ (d) $2^{x/(\ln 2)}$
4. (a) $(\ln 56)/(\ln 4)$ (b) $(\ln 50)/(\ln 3)$
   (c) $(-\ln 12)/(\ln 5)$ (d) $(\ln 49)/(\ln 2)$
   (e) $(\ln 3)/(\ln 7)$ (f) $(\ln (21/13))/(\ln 11)$
5. (a) $\ln 4$ (b) −2
   (c) $-25 \ln 5$ (d) $(5/6) \ln 10$
   (e) $10 \ln 2$ (f) $25 \ln 2$
6. (a) e^2 (b) $(1 - e^4)/2$
   (c) e^3 (d) $e^4 - 1$
7. (a) Vertical Asymptote: $x = 0$



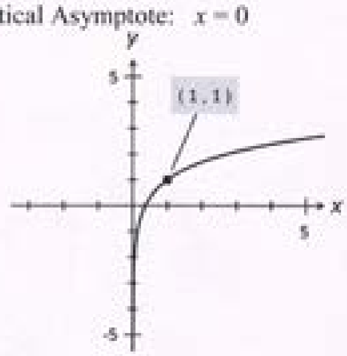
   (b) Vertical Asymptote: $x = 1$



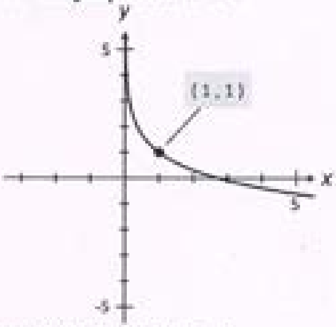
   (c) Vertical Asymptote: $x = -2$



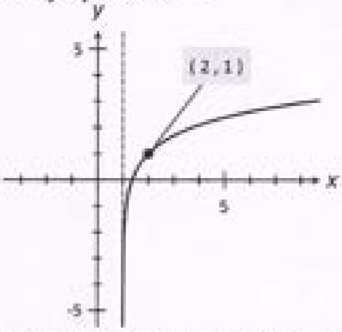
   (d) Vertical Asymptote: $x = 0$



7. (e) Vertical Asymptote: $x = 0$



(f) Vertical Asymptote: $x = 1$

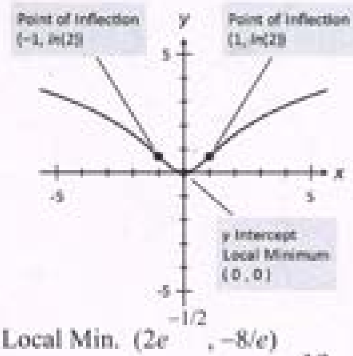


8. (a) $y = 2\ln(x+1)$ (b) $y = 2\ln(x) - 1$
   (c) $y = \ln(2-x)$ (d) $y = -\ln(1-x)$
9. (a) $-(\ln 200)/(\ln 2)$ (b) $-[(\ln 30)/(\ln 3)]/2$
   (c) $-1 - (\ln 80)/(\ln 5)$
   (d) $(\ln 10)/(\ln 125)$ (e) 2
   (f) $100(\ln 1.2)/(\ln 3)$
   (g) $(\ln(3/2))/(\ln 2)$
   (h) $(\ln(4/15))/(\ln(5/3))$
   (i) $(\ln(40/3))/(\ln 192)$
10. (a) $-\ln 2$, $\ln 2$ (b) $\ln 3$ (c) 3
   (d) $(\ln(3/2))/(\ln 3)$, $(\ln(4/3))/(\ln 3)$

Exercise 14.2

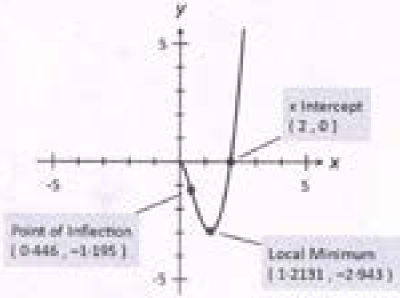
1. (a) $1/x$ (b) $1/x$
   (c) $1/(x+2)$ (d) $2/(2x+1)$
   (e) $8x/(1+x^2)$ (f) $-x/(4-x^2)$
   (g) $1/(x-2)$ (h) $12x^2/(x^3+1)$
2. (a) $1/(2x)$ (b) $-1/x$
   (c) $-4/(1-x)$ (d) $6/(1+2x)$
   (e) $1/[2(1+x)]$ (f) $-1/(1+x)$
   (g) $x/(1+x^2)$ (h) $-6/(1+2x)$
3. (a) $1/(x+1) + 1/(x-1)$
   (b) $3/(x-1) + 1/(x+1)$
   (c) $2x/(x^2+1) + 4/(x-1)$
   (d) $1/(x+1) + 1/[2(x-1)]$
   (e) $1/(x+2) - 1/(x-3)$
   (f) $1/(x+1) - 2x/(x^2+1)$
   (g) $4/(2x+1) - 1/(x-1)$
   (h) $1/[2(x+1)] - 3/x$

4. (a) $3t^2 \ln(1+t) + t^3/(1+t)$
   (b) $4t\ln(1-t) - 2t^2/(1-t)$
   (c) $\ln(1+t^2) + 2t^2/(1+t^2)$
   (d) $\ln(1-2t) - 2(1+t)/(1-2t)$
   (e) $[\ln(1-t) + t/(1-t)]/[\ln(1-t)]^2$ or
   $[(1-t)\ln(1-t) + t]/\{(1-t)[\ln(1-t)]^2\}$
   (f) $[2t\ln(t) - (1/t)(1+t^2)]/[\ln(t)]^2$
   or $[2t^2\ln(t) - (1+t^2)]/\{t[\ln(t)]^2\}$
5. (a) $\frac{1}{2}t^{-1/2}\ln(1+3t) + 3t^{1/2}/(1+3t)$
   (b) $3\ln(1+t) + 3t/(1+t)$
   (c) $t\ln(1-t) - t^2/[2(1-t)]$
   (d) $[1 - \ln(1+t)]/(t+1)^2$
6. (a) $-e^x/(1-e^x)$ (b) $-2e^x/(1-e^x)$
   (c) $e^x/(1+e^x) - 2e^{2x}/(1-e^{2x})$
   (d) $e^x/(1+e^x) + e^x/(1-e^x)$
   (e) $-e^{-x}\ln(x) - (1/x)e^{-x}$
   (f) $e^x\ln(1+x) + (1+e^x)/(1+x)$
7. (a) $2x\ln[\sin(x)] + x^2\cos(x)/\sin(x)$
   (b) $\cos(x)[1 + \ln(\sin(x))]$
   (c) $-\sin(t)\ln(\sin(t)) + \cos^2(t)/\sin(t)$
   (d) $\ln(\tan 2t) + t(2\sec^2 2t)/(\tan 2t)$
   (e) $\cos(t)[1 - \ln(\sin t)]/\sin^2(t)$
   (f) $e^{\sin x}(\cos x)(\ln\cos x) - (\sin x)e^{\sin x}/(\cos x)$
8. $y = x + 1$ 9. $y = e^2x - e^2$
10. $y = x/2 - 1/2 + \ln 2$, $y = -2x + 2 + \ln 2$
11. $(4, \ln 2)$ 12. $(0, \ln 2)$
13. Local Max. $(5, 5\ln 5 - 5)$
14. (a) (i) Local Min. (0, 0)
   (ii) Oblique inflection points $(\pm 1, \ln 2)$
   (b)



15. (a) (i) Local Min. $(2e^{-1/2}, -8/e)$
   (ii) Oblique inflection point $(2/e^{3/2}, -24/e^3)$

15. (b)



16. (a) \$200 000 per year
    (b) (i) After 4 months.
        (ii) After 33 months.
    (c) \$73 576 at $t = 4$ months

17. (a) 0 ms^{-1}, 1 ms^{-2} (b) $1/e^2$ ms^{-2}
    (c) $(ln\ 11)/10$ ms^{-2} (d) As $t \to \infty$, $a \to 0$.

18. (a) −0.35 m (b) 0.79 seconds
    (c) 3.64 m (d) 3.99 m

Exercise 14.3

1. (a) $2ln|x| + C$ (b) $(3/4)\ ln|x| + C$
   (c) $x + ln|x| + C$ (d) $(1/2)ln|3 + 2x| + C$
   (e) $x^2/2 + 2x + ln|x| + C$
   (f) $x + 2ln|x| - 1/x + C$
   (g) $(-1/5)\ ln|1 - 5x^2| + C$
   (h) $(-2/3)ln|2x^3 - 5| + C$
2. (a) $(1/2)ln|x^2 + 6x| + C$
   (b) $(1/2)\ ln|2x^3 + x^2 - 2x| + C$
   (c) $ln|1 + \sqrt{x}| + C$
   (d) $x^3/3 + 2ln|x| - 1/(3x^3) + C$
   (e) $-ln|e^x + e^{-x}| + C$
   (f) $(-5/2)\ ln|1 + e^{-2x}| + C$
3. (a) $-ln|\cos t| + C$
   (b) $(-1/2)\ ln|\cos(1 + 2x)| + C$
   (c) $ln|\cos(1 - x)| + C$
   (d) $(1/3)ln|\sin 3x| + C$
4. (a) $(1/\pi)\ ln|1 + \sin(\pi x)| + C$
   (b) $(1/\pi)\ ln\ |2 - \cos(\pi x)| + C$
   (c) $(1/2)\ ln\ |2x + \sin(2x)| + C$
   (d) $-ln\ |\sin(t) + \cos(t)| + C$
5. $ln|(x + 1)/(x + 2)| + C$
6. $A = -1/3$, $B = 4/3$;
   $(-1/3)ln|x + 1| + (4/3)ln|x - 2| + C$
7. $y = 2x^2 - (1/2)\ \ln|1 - 2x| + 4$
8. $y = e^x - x + (1/2)\ \ln|1 + x^2| - 3$

9. (a) $3/2 - ln\ 2$ (b) $1/2 + ln\ 2$
   (c) $1/2 + ln\ 2$ (d) 0.3879
   (e) 2.0235 (f) 2.6806
10. $ln\ 5$

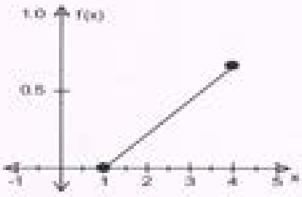
Exercise 15.1

1. (a)110/150 (b) 73/110
   (c) 311/375 (d) 138.0
   (e) 130.1, 258.3156
2. (a) 0.3125, 0.425, 0.168 75, 0.625, 0.031 25
   (b) 48 (c) 63.7
3. (a) 269/279 (b) 118
   (c) High possibility as $P(X > 125) = 14/269$ which is very low for the stage of the closed season.
4. (a) 58.25, 8.2576 (b) 73/100
   (c) 52.5
5. (a) 228.9024, 7033.5512
   (b) 217.86 (c) 343.125
6. (a) 241/750 (b) 6.1
   (c) 9.5

Exercise 15.2

1. (a) Yes; $f(x) \geq 0$ within given interval and area under curve is 1.
   (b) No; area under curve is not 1
   (c) Yes; $f(x) \geq 0$ within given interval and area under curve is 1.
   (d) No; area under curve is not 1
   (e) No; $f(x) < 0$, $-1 < x \leq 2$
   (f) Yes; $f(x) \geq 0$ within given interval and area under curve is 1.
2. (a) 2.5
   (b) $f(x) = 0.8 - 0.32x$ for $0 \leq x \leq 2.5$
   (c) 0.25 (d) 0.0625 (e) 0.25
3. (a) 2.5
   (b) $f(x) = \begin{cases} 0.4 - 0.16x & 0 \leq x < 2.5 \\ -0.4 + 0.16x & 2.5 \leq x \leq 5 \end{cases}$
   (c) 0.625 (d) 0.4688 (e) 0.2
4. (a) 0.2
   (b) $f(x) = \begin{cases} 0.04x & 0 \leq x < 5 \\ 0.4 - 0.04x & 5 \leq x \leq 10 \end{cases}$
   (c) 6.4645
5. (a) $\frac{4}{3}$
   (b) $f(x) = \begin{cases} 0.5 & 0 \leq x < \frac{4}{3} \\ 1 - 0.375x & \frac{4}{3} \leq x \leq \frac{8}{3} \end{cases}$
   (c) 1.8234
6. (a) 0.75 (b) 0.99 (c) $0.\overline{75}$ (d) 2/3, 1/18
7. (a) $\frac{3}{8}$ (b) $x^3/8$ (c) 1.5874 (d) 3/2, 3/20
8. (a) $b = 4$ (b) 0.75 (c) $\frac{2}{3}$ (d) 5/2, 13/12
9. (a) $a = 5$ (b) 0.625
   (c) 0.6 (d) 1 or 3

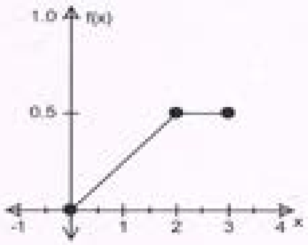
10. (a) $a = \frac{2}{9}$, $b = \frac{-2}{9}$
    (b)



    (c) 2.5   (d) 3, 1/2

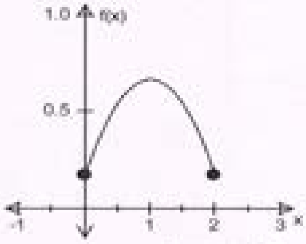
11. (a) 0.245   (b) 0.875
    (c) 0.1371   (d) 1.2929

12. (a) 1/4
    (b)



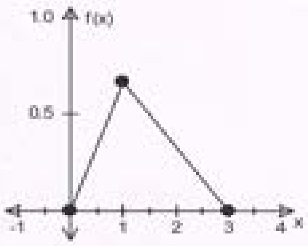
    (c) $(3 - x)/2$ (d) 2

13. (a) $a = 2/3$
    (b)



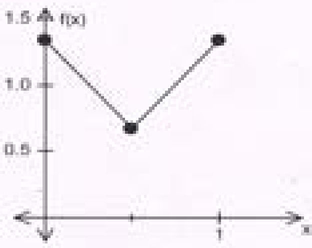
    (c) 1   (d) 1, 11/45

14. (a) 2/3
    (b)



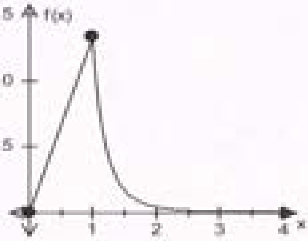
    (c) 3/5   (d) 0.2500

15. (a) 2/3
    (b)



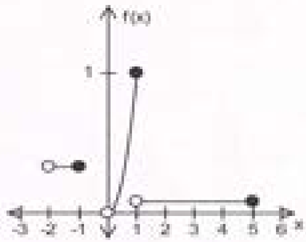
    (c) 0.2941   (d) 0.5, 0.5   (e) 0.5608

16. (a) 4/3
    (b)



    (c) 2/5   (d) $\sqrt{3}/2$ (e) 0.8418

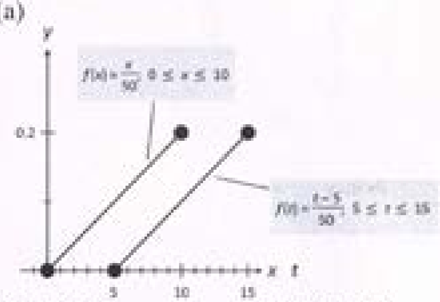
17. (a)



    (b) 2/3   (c) 11/12   (d) 7/8

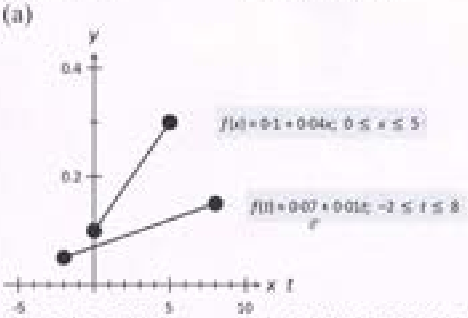
Exercise 15.3

1. (a)



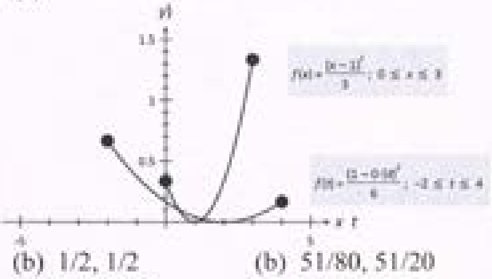
    (b) 1/4, 1/4   (c) 20/3, 35/3

2. (a)



    (b) 2/11, 2/11   (c) 275/144, 275/36

3. (a)



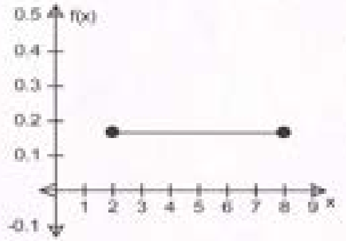
    (b) 1/2, 1/2   (b) 51/80, 51/20

4. (a) 0, $(\sqrt{5})/5$ (b) 0, 1

5. (a) 0, $(\sqrt{2})/2$ (b) 0, 1

Exercise 16.1

1. (a) $f(x) = \frac{1}{6}$ for $2 \le x \le 8$
   (b)



   (c) $\frac{1}{3}$ (d) $\frac{1}{2}$
2. (a) $f(x) = \frac{1}{4a}$ for $a \le x \le 5a$
   (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) $3a$, $4a^2/3$
3. (a) 5 (b) 0.75 (c) 1 (d) 3, 4/3
4. (a) $a = 1, b = 11$ (b) 3.5
   (c) 8.5 (d) 0
5. (a) $f(x) = 1$ $8 \le x \le 9$
   (b) 0.75 (c) 0.4
   (d) (i) 0.0256 (ii) 0.8704
6. (a) $f(x) = \frac{1}{2}$ $0 < x < 2$
   (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) 0.2461
7. (a) $f(v) = \frac{1}{7}$ $745 \le v \le 752$; 748.5, 2.0207
   (b) (i) 0 (ii) $\frac{5}{7}$ (c) 0.8672
8. (a) $f(t) = \frac{1}{30}$ $10 \le t \le 40$
   (b) (i) 0 (ii) $\frac{2}{3}$ (c) 25, $5\sqrt{3}$ (d) 0.7407
9. (a) $f(t) = \frac{1}{120}$ $0 < t \le 120$
   (b) 0 (c) 0.5 (d) 0.3633
10. (a) $f(a) = 0.079\,58$ $307.9075 \le a \le 320.4739$
    (b) 0.1665 (c) 0.4209
11. (a) 0, 1 (b) $(\sqrt{3})/3$ (c) $(\sqrt{3})/3$
12. (a) 0, 1 (b) $(1 + \sqrt{3})/4$ (c) $(1 + \sqrt{3})/4$

Exercise 17.1

1. (a) 0.9750 (b) 0.04947 (c) 0.1357
   (d) 0.9332 (e) 0.2858 (f) 0.2417
   (g) 0.8784 (h) 0.2255 (i) 0.7583
   (j) 0.7142
2. (a) 0.9906 (b) 0.03438 (c) 0.02743
   (d) 0.9693 (e) 0.07160 (f) 0.6662
3. (a) 0.9265 (b) 0.1057 (c) 0.7011
   (d) 0.01684
4. (a) 0.04978 (b) 0.7327 (c) 0.9744
   (d) 0.5892
5. (a) 2.15 (b) 0.56 (c) 2.09
   (d) 1.47 (e) 0.57 (f) 2.19

Exercise 17.2

1. (a) $k = 1$; 0.1587 (b) $k = 3$; 0.00135
2. (a) $k = 1$; 0.8414 (b) $k = -1$; 0.8414
3. (a) 0.3503 (b) 0.8757 (c) 0.2580
4. (a) 0.3437 (b) 0.04550 (c) 0.9545
5. (a) −1.16 (b) −0.17 (c) 1.13
   (d) −0.87 (e) 1.32 (f) 0.60
6. (a) 96.45 (b) 63.55 (c) 86.75
   (d) 67.18 (e) 6.74 (f) 28.07
7. (a) 25 (b) 10
8. (a) 12 (b) 13
9. (a) $\mu = 60, \sigma = 11$
   (b) $\mu = 150, \sigma = 34$

Exercise 17.3

1. (a) 0.5362 (b) 0.5721 (c) 0.2021
2. (a) 0.7475 (b) 0.6563 (c) 0.8780
3. (a) 0.00135 (b) 0.00135 (c) 0.05934
4. (a) 97 (b) $79 \le$ marks < 97
   (c) 528
5. (a) 0.62% (b) $\sigma = 97.1$
6. (a) 0.9176 (b) 0.8351 (c) 0.9101
   (d) 0.8750
7. (a) 25.25% (b) 50.50% (c) 0.02972
   (d) 9.12% (e) 11.5%
8. (a) 79 (b) 54 (c) 2.88 cm
   (d) 0.1499
9. (a) 0.1587 (b) 0.1587 (c) 0.8413
   (d) 0.8114 (e) 0.14997
10. (a) 0.39 (b) 0.16 (c) 0.10
11. (a) 26.41 (b) 36.62 (c) 38.14
12. (a) $\mu = 95, \sigma = 8$ (b) $\mu = 150, \sigma = 6$
13. (a) $\mu = 57.32, \sigma = 13.42$
    (b) $\mu = 48, \sigma = 16.198$
14. (a) (i) 0.3086 (ii) 0.5328 (iii) 0.9799
    (b) 102.56 (c) 0.3489
15. (a) 0.9104 (b) 0.8991
    (c) 11.3 (d) 0.008 017

Exercise 18.1

1. Self-selection sampling and hence non-random.
   Possible bias:
   - Those without internet access are excluded.
   - Possibility of "full-scale" assault by interest groups distorting results of poll.
   - Responses generated by "robots".
   - Multiple responses from individuals.
   - Non-English speakers would probably be excluded.
2. Self-selection sampling and hence non-random.
   Possible bias:
   - Those without twitter accounts are excluded.
   - Possibility of "full-scale" assault by interest groups distorting results of poll.
   - Responses generated by "robots".
   - Multiple responses from individuals.
3. Self-selection sampling, hence non-random.
   Possible bias:
   - Those without mobile phones are excluded.
   - Possibility of "full-scale" assault by interest groups distorting results of poll.
   - Responses generated by "robots".
   - Multiple responses from individuals.

4. Self-selection sampling and hence non-random.
   Possible bias:
   - Those without Facebook accounts are excluded.
   - Possibility of "full-scale" assault by interest groups distorting results of poll.
   - Responses generated by "robots".
   - Multiple responses from individuals.
5. Method of sampling is non-random.
   Possible bias:
   - Depending on distribution of houses on the sides fronting the street, sample may have more houses on one side of the street.
   - No allowances made for "building types" e.g. duplexes and units. For example: if a block of flats in the street has 80 homes, then 8 of these homes would have been selected.
   - No allowances made for presence of homes situated near parks or other structures along the street.

   Reduction of bias:
   - Consider distribution of homes on sides of street.
   - Types of dwellings fronting the street.
   - Consider location of dwellings and adjacent structures.
   - Consider age and gender of respondents.
6. A form of convenience sampling and hence non-random.
   Possible bias:
   - Homes without phones excluded.
   - Non-English speakers could be excluded if interview conducted only in English.
   - Residents in the rural and smaller towns excluded and hence poll result would be biased towards the city residents
7. A form of convenience sampling and hence non-random.
   Possible bias:
   - Times the homes were called. Successful calls made during "normal" working hours would poll only "stay at home mums".
   - If calls made to mobile phones during working hours may not get reliable answers – interrupting work etc.
   - Non-English speakers could be excluded if interview conducted only in English.
   - Need to consider "categories" of parents: e.g. both working, single parents etc.
8. Stratified sampling would be recommended. Sample to comprise the following number of students from the three year groups.

Year	Number of students	No of students in sample
10	380	36
11	320	30
12	360	34
Total	1060	100

8. Each year group to be further divided into male and female students and the representatives from the year groups to reflect the distribution of male and female students in each year group.
9. Stratified sampling would be best.
   Students in sample to reflect actual:
   - proportion of girls and boys
   - proportion of students in inner and outer suburbs
   - proportion of students in government and private schools
10. Stratified sampling would be best.
    - Participants in sample to reflect actual proportion of residents in various states
    - Participants from each state to represent proportion of urban and non-urban residents.
    - Participants to reflect ethnic composition of state/country.
    - Participants to reflect proportion of males and females in state and country.
    - Participants to reflect the various age groups.

Example 19.1

1. (a) Approx. normal $\mu = 0.38$, $\sigma = 0.054\ 268$
   (b) 0.356 23 (c) 0.396 75
   (d) 0.084 82
2. (a) Approx. normal $\mu = 0.09$, $\sigma = 0.020\ 236$
   (b) 0.31 059 (c) 0.940 86
   (d) 0.083 19
3. (a) Approx. normal $\mu = 0.47$, $\sigma = 0.070\ 583$
   (b) 0.167 88
4. (a) Approx. normal $\mu = 0.63$, $\sigma = 0.039\ 421$
   (b) 0.423 85
5. (a) 1/6
   (b) (i) Approx. normal $\mu = 1/6$, $\sigma = 0.058\ 926$
   (ii) 0.467 49
6. (a) 2/5
   (b) (i) Approx. normal $\mu = 2/5$, $\sigma = 0.063\ 246$
   (ii) 0.113 85
7. (a) 0.908 79
   (b) (i) Approx. normal $\mu = 0.908\ 79$, $\sigma = 0.028\ 791$
   (ii) 0.612 95
8. (a) 0.773 37
   (b) (i) Approx. normal $\mu = 0.773\ 37$, $\sigma = 0.029\ 603$
   (ii) 0.736 55
9. $n \geq 69$ 10. $n \geq 16$
11. (a) 12.5 minutes
    (b) Approx. normal $\mu = 0.5$, $\sigma = 0.079\ 057$
    (c) (i) 1/3 (ii) 0.897 05
12. (a) Approx. normal $\mu = 0.6$, $\sigma = (\sqrt{6})/50$
    (b) 0.419 13 (c) $n \geq 96$
13. (a) $X \sim B(8, 1/8)$
    $\hat{\pi} \sim N(0.067\ 347, 0.041\ 770^2)$
    (b) 0.070 70 (c) 0.048 60

14. (a) $p(x) = \dfrac{\binom{7}{x}\binom{5}{4-x}}{\binom{12}{4}}$ for $x = 0, 1, 2, 3, 4$

$\hat{\pi} \sim N(\mu = 28/33,\ \sigma = (\sqrt{35})/99)$

(b) 0.202 10 (c) 0.065 565

15. 0.48 or 0.52

Exercise 20.1

1. (a) {6/10, 5/10, 6/10, 8/10, 6/10}
   (b) (i) 6/10, $(\sqrt{15})/25$
   (ii) {0, −0.6325, 0, 1.5811, 0}
2. (a) 31/50
   (b) (i) Approx. normal $\mu = 31/50$, $\sigma = (\sqrt{4178})/500$
   (ii) 0 (iii) N(0, 1)
3. (a) {8/10, 8/10, 4/10, 6/10, 5/10}
   (b) (i) 8/10, $(\sqrt{10})/25$
   (ii) {0, 0, −2.5820, −1.2910, −1.8974}
4. (a) {17/30, 10/30, 13/30, 13/30, 20/30, 14/30, 13/30, 17/30, 13/30, 18/30}
   (b) (i) Approx. normal $\mu = 14/30$ $\sigma = (2\sqrt{105})/225$
   (ii) {1.1053, −1.5492, −0.3684, −0.3684, 2.3238, 0, −0.3684, 1.1053, −0.3684, 1.4907}
   mean = 0.3002, s.d. = 1.1003
5. (a) 59/200
   (b) Approx. normal $\mu = 59/200$ $\sigma = 0.032247$
   (c) Approx. normal $\mu = 59/200$ $\sigma = 0.032247$
6. (a) Approx. normal $\mu = 0.32$, $\sigma = 0.065\ 97$
   (b) N(0, 1)

Exercise 20.2

1. 1/5
2. (a) 1/3
   (b) Approx. normal $\mu = 1/3$, $\sigma = (\sqrt{30})/90$
3. (a) 21/100
   (b) $0.15 \le \pi \le 0.27$ (c) $n \ge 64$
4. (a) 17/30
   (b) $0.49 \le \pi \le 0.64$ (c) $n \ge 104$
5. (a) 4/5
   (b) Not likely to be a fair estimate as only urban voters were sampled.
   (c) $0.76 \le \pi \le 0.84$
6. (a) 2/5
   (b) $2/5 - (3\sqrt{6})/100 \le \pi \le 2/5 + (3\sqrt{6})/100$
7. $n = 312$ 8. $n = 337$
9. 92.3% 10. 99.2%
11. (a) 300 (b) 16 or 17
12. (a) $0.63 \le \pi \le 0.67$; $0.62 \le \pi \le 0.68$ $0.61 \le \pi \le 0.69$
    (b) Second sample less in favour. Proportion in second sample below confidence intervals for π.
13. (a) $0.53 \le \pi \le 0.69$; $0.51 \le \pi \le 0.71$ $0.48 \le \pi \le 0.74$
    (b) Second sample showed better performance. Proportion in second sample above confidence intervals for π.
    But premature to conclude that the nap was the cause of the better performance, could be due to other factors not yet identified.

Exercise 20.3

1. (a) Not significantly different as $\hat{p}$ lies outside the critical region.
   (b) Not significantly different as $\hat{p}$ lies outside the critical region.
   (c) Not significantly different as $\hat{p}$ lies outside the critical region.
2. (a) Significantly different as $\hat{p}$ lies inside the critical region.
   (b) Significantly different as $\hat{p}$ lies inside the critical region.
   (c) Not significantly different as $\hat{p}$ lies outside the critical region.
3. (a) Significantly different as $\hat{p}$ lies inside the critical region.
   (b) Not significantly different as $\hat{p}$ lies outside the critical region.
4. (a) Not significantly different as $\hat{p}$ lies outside the critical region.
   (b) 19.25%
5. (a) 12.1% (b) $n \ge 801$
6. (a) 15.5% (b) $n \ge 3942$